Sound Transmission Loss of One-Dimensional Systems Containing Fictitious Metamaterials

Waldemar Maysenhölder

Fraunhofer-Institut für Bauphysik, 70569 Stuttgart, E-Mail: waldemar.maysenhoelder@ibp-extern.fraunhofer.de

Introduction

Metamaterials are distinguished from 'normal' materials by unusual properties. Some behave as if the mass density or some elastic modulus were negative. However, in 'real metamaterials' such unusual properties appear only in limited frequency ranges. Moreover, these properties are as a rule complex-valued, i.e. implicate damping, and depend on frequency. By contrast, the properties of 'fictitious metamaterials' may be defined arbitrarily, regardless of physical realizations.

Although such 'exercises' with fictitious material properties may be regarded as academic, they can be useful for the development of practical real metamaterials, be it with respect to theoretical limits, understanding or inspiration. Negative masses or mass densities were not only beneficial for building acoustics or other sound transmission problems. They were also advantageous even for the detection of gravitational waves by laser interferometry: If one of the two elastically mounted mirrors of an interferometer had a negative mass, the quantum-mechanical noise caused by the impact of the photons were then eliminated [1]. First experiments already show that the idea works in principle. This may be taken as encouragement for further pursuing high aims also in acoustics, possibly including active solutions.

The claim of the present contribution is much more modest, though. After a brief description of the one-dimensional model four basic types of partitions are studied: Mass, Mass – Spring – Mass, Elastic Layer, and finally Mass – Elastic Layer – Mass. The elastic layer may be considered as a generalization of both a rigid mass and a massless spring.

Model

The calculation of the sound transmission loss R of a partition between two fluid half-spaces (Fig. 1) is carried out in a one-dimensional model using transfer matrices [2]. The lossless fluids in the sending and receiving regions I and III are characterized by their mass densities, sound speeds, the ratio of the their impedances being denoted by ζ.

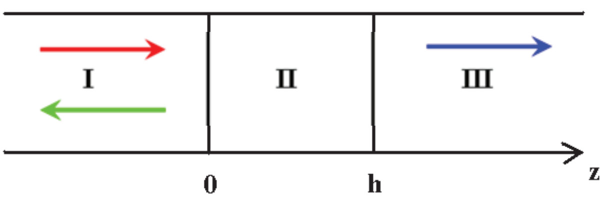

Fig. 1: One-dimensional sound transmission model with incident and reflected waves in region I, partition with thickness h in region II, and transmitted wave in region III.

The transfer matrix of the partition determines the transmission coefficient $\mathscr{T}$, from which the transmission factor τ and the transmission loss R follow:

$$R = -10\lg\tau\,, \qquad \tau = \frac{1}{\zeta}\left|\mathscr{T}\right|^2\,, \qquad \zeta = \frac{\rho_{\mathrm{III}}\,c_{\mathrm{III}}}{\rho_{\mathrm{I}}\,c_{\mathrm{I}}} = \frac{Z_{\mathrm{III}}}{Z_{\mathrm{I}}}\,. \tag{1}$$

The normalization of frequency ω, masses M, spring stiffness C and impedances Z used in the next two sections on partitions consisting of a mass only and a mass-spring-mass system, respectively, is denoted by a hat on the symbol. For a detailed description of the matter see [2].

Mass

Can the usual mass law for positive masses be also used for negative masses? The answer is yes; but for complex masses the more general form

$$\tau = \frac{4\zeta}{(\zeta+1)^2 + 2(\zeta+1)\dfrac{\operatorname{Im}\hat{M}}{\hat{Z}_{\mathrm{I}}}\hat{\omega} + \dfrac{|\hat{M}|^2}{\hat{Z}_{\mathrm{I}}^2}\hat{\omega}^2} \xrightarrow{\text{real }\hat{M}} \frac{4\zeta}{(\zeta+1)^2 + \left(\hat{\omega}\dfrac{\hat{M}}{\hat{Z}_{\mathrm{I}}}\right)^2} \tag{2}$$

is needed. The imaginary part of the complex mass implies energy dissipation.

As an example the transmission loss of a harmonic oscillator HO (a rigid mass attached via a spring to the 'duct walls' in Fig. 1 with some damping) is determined by the mass law Eq. (2a), since in this case the HO can be fully described by a complex, frequency-dependent mass (Fig. 2).

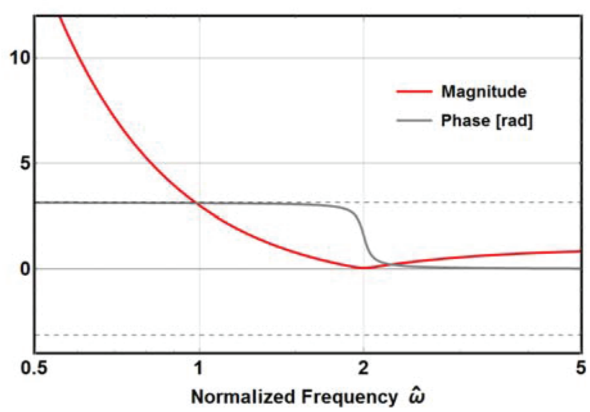

Fig. 2: Magnitude and phase of the normalized complex mass representing a harmonic oscillator.

It is strongly negative at low frequencies (physically speaking: this is the spring-controlled region) and becomes positive above its resonance frequency $\hat{\omega}_{\mathrm{HO}} = 2$, where it attains a minimum.

The transmission loss of the HO with mass according to Fig. 2 is shown in Fig. 3 together with the curve for another HO with zero resonance frequency and the usual mass law, which is obtained by additionally setting the damping to zero.

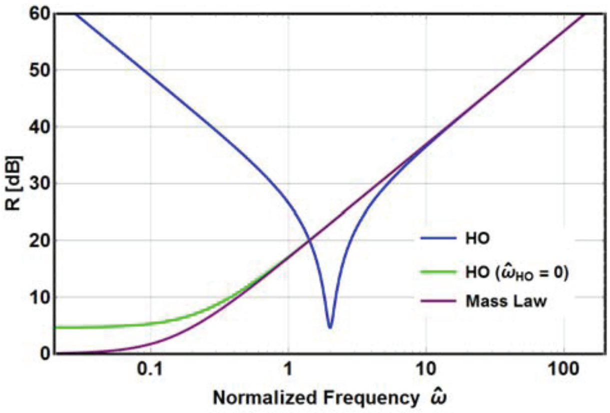

Fig. 3: Transmission loss of harmonic oscillators: with complex mass as in Fig. 2 (blue), with resonance frequency set to zero (green), and with damping set to zero as well, which leads to the usual mass law (purple).

Mass – Spring – Mass

For a partition consisting of two rigid masses M_1 and M_2 and a massless spring in between with stiffness C the transmission factor

$$\tau = \frac{4\hat{Z}_{\mathrm{I}}\,\hat{Z}_{\mathrm{III}}}{\sum_{n=0}^{6} Q_n\,\hat{\omega}^n} \tag{3}$$

is inversely proportional to a sixth-degree polynomial in the frequency. The high-frequency limit is determined by the sixth power of the frequency leading to the familiar slope of the transmission loss of 18 dB per octave. The prefactor reads

$$Q_6 = \left|\hat{M}_1\,\hat{C}\,\hat{M}_2\right|^2 . \tag{4}$$

Apart from the limiting transmission behavior at low and high frequencies the minima and maxima are of particular interest. In case of real, i.e. positive or negative, masses and stiffness they are readily obtained – after the separation of the maximum transmission at zero frequency – from a quadratic equation for the frequency squared [2]. For the symmetric case $\hat{Z}_{\mathrm{I}} = \hat{Z}_{\mathrm{III}} = \hat{Z}$, $\hat{M}_1 = \hat{M}_2 = 1/2$ and $\hat{C} = 1$ with a resonance frequency in vacuum of $\hat{\omega}_{\mathrm{res\_free}} = 2$ one arrives at the simple results

$$\hat{\omega}_{\min} = \frac{1}{\sqrt{3}}\hat{\omega}_{\max}, \qquad \tau_{\min} = \frac{27\hat{Z}^2}{\left(4-\hat{Z}^2\right)\left(1+2\hat{Z}^2\right)^2}, \tag{5}$$

$$\hat{\omega}_{\max} = 2\sqrt{1-\hat{Z}^2} > 0, \quad \tau_{\max} = 1 . \tag{6}$$

If the fluid impedance is large enough, e.g. with some partition in water, there is no extremum at all. The transmission loss for this symmetric case with $\hat{Z} = \hat{Z}_{\mathrm{air}} \approx 0.071$ is shown in Fig. 4 in blue with total transmission at $\hat{\omega}_{\max} \approx 2$. This is the behavior typical of a double wall, e.g. made of plasterboards. The green curve shows the case with one of the masses negative. Since the sum of the two masses is zero, the transmission loss is close to zero in the mass-law region below $\hat{\omega}_{\max} = 0.5$. The highest transmission loss at all frequencies is obtained with both masses negative. In the high-frequency limit all three curves coincide because of Eq. (4).

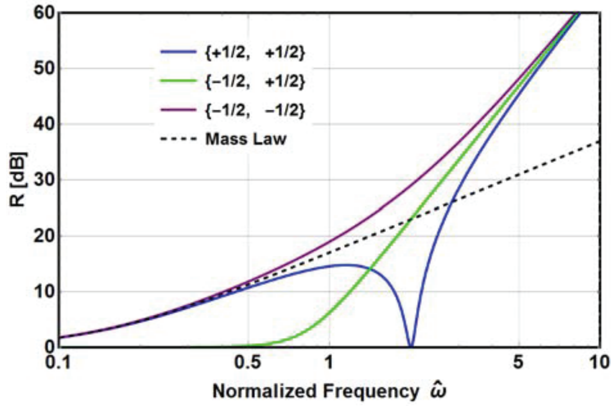

Fig. 4: Transmission loss of mass-spring-mass systems in air with stiffness $\hat{C} = 1$ and masses of equal magnitude $\left|\hat{M}_1\right| = \left|\hat{M}_2\right| = 1/2$, but with different sign combinations. Dotted line: mass law for $\left|\hat{M}\right| = 1$.

Without losses the coefficients Q_n are invariant under simultaneous sign reversal. Therefore, the 'optimum curve' in Fig. 4 represents also the case of two positive masses and negative stiffness.

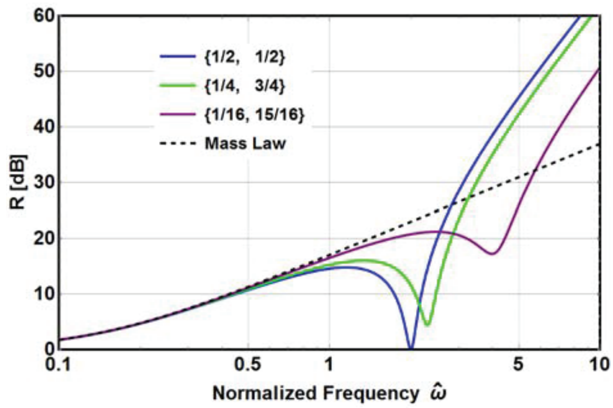

Fig. 5: As Fig. 4, but with only positive masses, including unequal ones.

Cases with masses of unequal magnitude are discussed in [2]. Here it must suffice to remind that if both masses are unequal and positive, but with constant sum $\hat{M}_1 + \hat{M}_2 = 1$, the transmission loss minimum lies above 0 dB and at higher frequencies than in the symmetric case (Fig. 5). Because of Eq. (4) all curves lie below the 'optimum curve' in Fig. 4.

Elastic Layer

In reality rigid masses and massless springs, which are useful idealizations for theoretical studies, do not exist. Every massive object has some compliance and every spring has some inertia. The elastic layer examined in the following can be regarded as a generalization of the two idealized objects. For one-dimensional considerations a layer is simply characterized by mass density ρ_L, elastic modulus K_L, and thickness h. Here, we assume no damping and no frequency dependence of ρ_L and K_L, which implies constant real ρ_L and constant real K_L. When one allows both positive and negative signs for compliance and inertia, four sign combinations arise. They are denoted by + +, + –, – +, – –, where the first sign refers to the density and the second to the elastic modulus.

Depending on these signs the sound speed c_L, the impedance Z_L and the normalized frequency φ_L,

$$c_\mathrm{L} = \sqrt{\frac{K_\mathrm{L}}{\rho_\mathrm{L}}}, \tag{7}$$

$$Z_\mathrm{L} = \rho_\mathrm{L} c_\mathrm{L} = \rho_\mathrm{L}\sqrt{\frac{K_\mathrm{L}}{\rho_\mathrm{L}}} \overset{?}{=} \sqrt{\rho_\mathrm{L} K_\mathrm{L}} \tag{8}$$

$$\varphi_\mathrm{L} = k_\mathrm{L} h = \frac{\omega}{c_\mathrm{L}} h \tag{9}$$

take real or imaginary values. Note that the equal sign with the question mark is valid only for positive density!

It turns out that, although the transmission coefficients $\mathcal{T}$ are different for the four sign combinations, the transmission factor τ depends only on the product of the two signs. With the abbreviation

$$\zeta_\mathrm{L} = \frac{|Z_\mathrm{L}|}{Z_\mathrm{I}} \tag{10}$$

corresponding to Eq. (1c) one arrives at

$$\tau_{++} = \tau_{--} = 4\left\{ \begin{array}{l} 2 + \left(\zeta^{-1} + \zeta\right) + \\ \left[\left(\dfrac{\zeta_\mathrm{L}^2}{\zeta} + \dfrac{\zeta}{\zeta_\mathrm{L}^2}\right) - \left(\zeta^{-1} + \zeta\right)\right] \sin^2\left(|k_\mathrm{L}| h\right) \end{array} \right\}^{-1} \tag{11}$$

for equal signs. Total transmission occurs under the following conditions: (i) for equal fluids $(\zeta = 1)$ if $\sin^2\left(|k| h\right) = 0$, and (ii) for unequal fluids $(\zeta \neq 1)$ if $\zeta_\mathrm{L}^2 = \zeta$ and $\sin^2\left(|k| h\right) = 1$. The second case is of special importance for ultrasound applications. The elastic layer with properties adapted to the impedances of regions I and III is called 'transmission plate' [3].

For unequal signs the sine in Eq. (11) changes to the hyperbolic sine:

$$\tau_{+-} = \tau_{-+} = 4\left\{ \begin{array}{l} 2 + \left(\zeta^{-1} + \zeta\right) + \\ \left[\left(\dfrac{\zeta_\mathrm{L}^2}{\zeta} + \dfrac{\zeta}{\zeta_\mathrm{L}^2}\right) + \left(\zeta^{-1} + \zeta\right)\right] \sinh^2\left(|k_\mathrm{L}| h\right) \end{array} \right\}^{-1}. \tag{12}$$

In addition, a sign change in the square bracket is noticed. For equal fluids $(\zeta = 1)$ there is total transmission only in the low-frequency limit $|k| h \to 0$. For unequal fluids $(\zeta \neq 1)$ there is never total transmission!

The transmission according to Eqs. (11) and (12) is exemplified in Fig. 6 for equal fluids $(\zeta = 1)$ and $\zeta_\mathrm{L} = 100$.

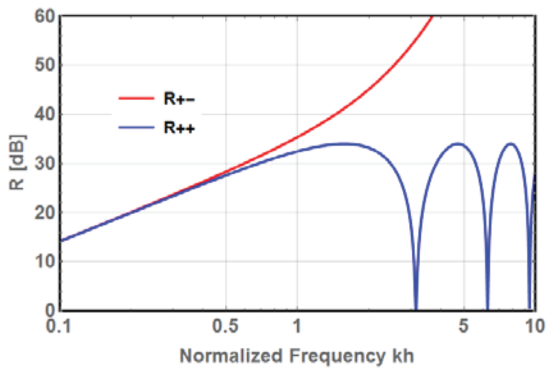

Fig. 6: Transmission loss of elastic layers with normalized impedance magnitude $\zeta_\mathrm{L} = 100$ between equal fluids. The signs of ρ_L and K_L are different for the curve R_{+-} and equal for the curve R_{++}.

Mass – Elastic Layer – Mass

Finally, an elastic layer is sandwiched between two (idealized rigid) masses. The transmission coefficient

$$\mathcal{T} = 2\left\{ \begin{array}{l} \left[\left(1 + \dfrac{Z_\mathrm{I}}{Z_\mathrm{III}}\right) - \mathrm{i}\dfrac{M_1 + M_2}{Z_\mathrm{III}}\,\omega\right] \cos\varphi_\mathrm{L} \\ -\mathrm{i}\dfrac{Z_\mathrm{L}}{Z_\mathrm{III}}\left[1 + \dfrac{\left(Z_\mathrm{I} - \mathrm{i}M_1\omega\right)\left(Z_\mathrm{III} - \mathrm{i}M_2\,\omega\right)}{Z_\mathrm{L}^2}\right] \sin\varphi_\mathrm{L} \end{array} \right\}^{-1} \tag{13}$$

– here written without normalization – leads to a fairly long transmission factor formula (which is definitely too long for the present two-column format).

Again no frequency dependence and no damping is assumed, i.e. M_1, ρ_L, K_L, M_2 are all constant and real, i.e. positive or negative. Running through all sign combinations amounts to $2^4 = 16$. While the transmission loss of the bare elastic layer depends only on the product of the signs of ρ_L and K_L, this is not (or only approximately) true for the sandwiched partition. This becomes clear by looking at the mass-law region at lower frequencies, where the sum of the masses of the three constituents is decisive for sound transmission, while the stiffness becomes unimportant. Here it certainly matters, whether the mass density of the layer is positive or negative.

In case of two masses with equal magnitude the number of essentially different configurations is reduced by symmetry. Eight curves are shown in Figs. 7 and 8 for a sandwich in air with $|M_1| = |M_2| = 10$ kg/m², $h_L = 8\,\mathrm{cm}$, $|\rho_L| = \rho_{air}$, and $|K_L| = K_{air}$ with a small damping loss factor $|\eta_L| = 0.01$ included.

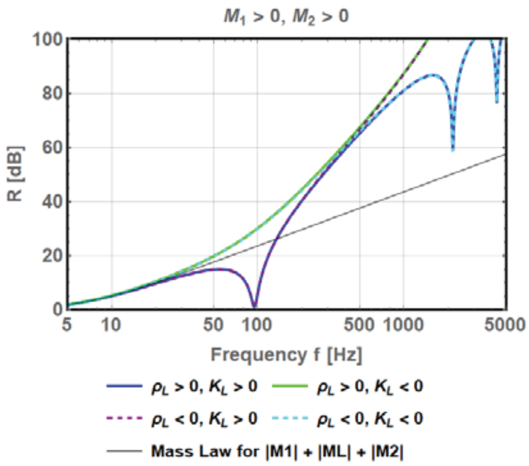

Fig. 7: Transmission loss of elastic layers sandwiched between two equal rigid positive masses for the four sign combinations of ρ_L and K_L. The black line shows the mass law as indicated in the legend.

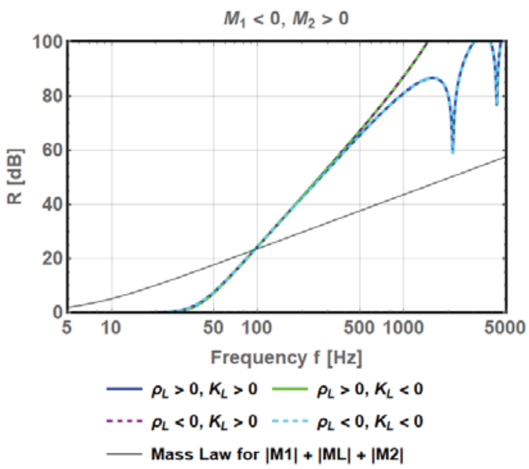

Fig. 8: As Fig. 7, but with two rigid masses of equal magnitude, but different sign.

This example is again inspired by a plasterboard double wall. By visual inspection one counts only four different curves. It seems that once more only the sign of the product $\rho_L K_L$ affects the transmission loss. However, this is only approximately the case, because the mass of the elastic layer, $|\rho_L| h = \rho_{air} h$, is small compared to the rigid masses. Therefore, the idealization of the air cavity as a massless spring is perfectly appropriate well below the cavity resonances (here at about 2 kHz). In this frequency range the curves behave in exactly the same way as in Fig. 2. The 'optimum curve' is recovered, too.

Conclusion

The sound transmission loss of basic elements (rigid mass, massless spring, elastic layer), which are often used for a simplified description of partitions, is examined in a one-dimensional transfer-matrix model. The focus is on the inclusion of metamaterial properties, which leads to remarkable effects and instructive findings. Firstly, the mass law is formulated for general complex masses. For a negative mass the familiar mass law still holds and gives the same result as for the positive sign, since only the mass squared enters the formula. In the sequel only frequency-independent properties with real, i.e. positive or negative, values are considered. The sound transmission loss of a mass-spring-mass system is optimal, if the spring constant is positive and both masses are equal and negative, or equivalently, if the spring constant is negative and both masses are equal and positive. The transmission loss of a homogeneous one-dimensional elastic layer, which may be regarded as a generalization of a rigid mass or a massless spring, shows a similar symmetry. With respect to the signs of the elastic modulus and the mass density, the transmission loss depends only on the product of the two signs, i.e. on whether the signs are equal or different. For different signs the sound speed becomes imaginary and there is never total transmission at finite frequencies. The elastic layer sandwiched between two rigid masses is a generalization of the mass-spring-mass partition and may be used for assessing the validity of both the massless-spring and the rigid-mass idealization.

Acknowledgement

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References

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