

Spatial Aliasing and Loudspeaker Directivity in Unified Wave Field Synthesis Theory

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Introduction

Sound Field Synthesis aims at the physical reproduction of arbitrary target sound fields over an extended listening area by employing a dense ensemble of loudspeakers, termed the *secondary source distribution (SSD)*. The loudspeakers are fed with properly chosen *driving functions*, so that the resultant field of the individual SSD elements ideally coincides with the target sound field in the intended receiving area [1, 2].

Wave Field Synthesis (WFS) is one of the prominent SFS methods, extracting the required driving functions from a suitable boundary integral representation of the target sound field [3, 4]. In a recent contribution by the present authors a unified WFS formulation was given, allowing the reproduction of arbitrary virtual sound fields, by using a smooth, convex distribution of secondary sources and optimizing the synthesis on an arbitrary reference curve [5].

Besides WFS-like implicit approaches explicit solutions are known for the SFS problem in special SSD geometries. For a linear loudspeaker array the explicit solution is referred to as the *Spectral Division Method (SDM)*, obtaining driving functions in the wavenumber domain [6]. Recently it was proven by the present authors that under high frequency conditions a wavefront matching, spatial approximation can be given for the general explicit solution, shown to be equivalent to the unified WFS driving functions [7].

Basic WFS and SDM theories assume the continuous distribution of 3D point sources as the secondary source distribution. In practice these assumptions are obviously violated, when an ensemble of directive sources located at discrete positions is applied. Both the artifacts and possibilities for their avoidance have been studied extensively in the related literature [8, 9, 10]. Discretization results in clearly audible artifacts in the synthesized sound field: amplitude errors and coloration of the synthesized field, as well as echoes following the intended synthesized wavefront due to *spatial aliasing*.

The present contribution revisits the topic of source directivity and spatial aliasing within the context of unified WFS theory.

Theoretical basics

The local wavenumber vector: Assume an arbitrary steady-state target/virtual sound field given by the general polar form $P(\mathbf{x}, \omega) = A^P(\mathbf{x}, \omega) e^{j\phi^P(\mathbf{x}, \omega)}$, with $A^P(\mathbf{x}, \omega), \phi^P(\mathbf{x}, \omega) \in \mathbb{R}$. The propagation dynamics of the sound field is described by its phase function, for which the local wavenumber vector $\mathbf{k}^P(\mathbf{x})$ can

be introduced: $\mathbf{k}^P(\mathbf{x}) = [k_x^P(\mathbf{x}), k_y^P(\mathbf{x}), k_z^P(\mathbf{x})]^T = -\nabla_{\mathbf{x}}\phi^P(\mathbf{x}, \omega)$. The local wavenumber vector is perpendicular to the wavefront in each position, thus pointing into the local propagation direction [5]. For simple sound fields—e.g. plane waves, spherical and cylindrical waves—the length of the local wavenumber vector is given by the local dispersion relation $|\mathbf{k}^P(\mathbf{x})| = \frac{\omega}{c} = k$, where c denotes the speed of sound and k is the acoustic wavenumber.

In the followings the horizontal reproduction of sound fields is discussed by applying a contour of loudspeakers located in the plane of synthesis, chosen to be the $z = 0$ plane. The synthesis is thus restricted to the reproduction of virtual sound fields propagating along this plane with $k_z^P(x, y, 0) \equiv 0$. In the plane of synthesis along with the local dispersion relation either k_x^P or k_y^P completely describes the local propagation direction of the virtual sound field.

The Stationary Phase Approximation: The backbone of both WFS theory and SDM in the spatial domain is formed by the stationary phase approximation (SPA). The SPA yields approximate asymptotic solutions of complex integrals of the form with $\mathbf{x} \in \mathbb{R}^n$

$$\int_{-\infty}^{\infty} F(x) e^{j\phi(x)} dx \approx \sqrt{\frac{2\pi}{|\phi''_{xx}(x^*)|}} F(x^*) e^{j\phi(x^*) \pm j\frac{\pi}{4}} \quad (1)$$

when $e^{j\phi(x)}$ is highly oscillating, $F(x)$ is comparably slowly varying and x^* is the *stationary point*, defined as $\phi'_x(x^*) = 0$ with ϕ'_x denoting the derivative of ϕ with respect to x .

The basic idea behind the SPA is that once $F(x)$ is a slowly varying smooth function, then the integral of rapid oscillation cancels out, and the greatest contribution to the total integral comes from the immediate surroundings of the stationary point.

2.5D Wave Field Synthesis: Assume a smooth convex SSD located at $\mathbf{x}_0 = [x_0, y_0, 0]^T$ consisting of a continuous distribution of 3D point sources, described by the 3D Green's function. In this geometry the synthesized field at a receiver position $\mathbf{x} = [x, y, 0]^T$ inside the area bounded by the SSD is described by the Kirchhoff approximation of 2.5D Kirchhoff-Helmholtz integral, from which the 2.5D driving functions can be extracted as [5]

$$D_{\text{WFS}}(\mathbf{x}_0, \omega) = w(\mathbf{x}_0) \sqrt{\frac{8\pi}{jk}} \sqrt{d_{\text{ref}}(\mathbf{x}_0)} \underbrace{j k_n^P(\mathbf{x}_0) P(\mathbf{x}_0, \omega)}_{\approx -\frac{\partial}{\partial n} P(\mathbf{x}, \omega)|_{\mathbf{x}=\mathbf{x}_0}} \quad (2)$$

where $k_n^P(\mathbf{x}_0)$ is the normal component of local wavenumber vector, $w(\mathbf{x}_0)$ is a window function selecting the active secondary elements and $d_{\text{ref}}(x_0)$ denotes the referencing

function as described in [5]. Here $\mathbf{x}_{\text{ref}}(x_0)$ describes the reference position for a given SSD element at \mathbf{x}_0 , where the implicit relation $\mathbf{k}^P(\mathbf{x}_0) = \mathbf{k}^G(\mathbf{x}_{\text{ref}}(\mathbf{x}_0) - \mathbf{x}_0)$ is satisfied on a prescribed *reference curve*.

The above driving functions ensure optimal synthesis on the reference curve within the validity of the stationary phase approximation, i.e. under high-frequency conditions.

2.5D Spectral Division Method: The explicit solution for a linear SSD exploits the fact that the synthesized field at an arbitrary receiver position is obtained in the form of a spatial convolution, corresponding to a spectral multiplication in the wavenumber domain [6]. Hence the spectrum of the required driving function is yielded by the ratio of the virtual field wavenumber content and the SSD elements' transfer function, for 3D point sources given by the Green's function. The spatial driving function is then obtained in the form of a spatial inverse Fourier transform

$$D_{\text{SDM}}(x_0, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{P}(k_x, y_{\text{ref}}, \omega)}{\underbrace{\tilde{G}(k_x, y_{\text{ref}}, \omega)}_{\tilde{D}(k_x, \omega)}} e^{-jk_x x_0} dk_x \quad (3)$$

with y_{ref} being a pre-defined reference line parallel with the SSD at which the synthesis is optimized.

By applying the stationary phase method to approximate both the forward and inverse Fourier transforms in equation (3) the SDM driving functions can be expressed entirely in the spatial domain. These driving functions are proven to be equivalent to the linear 2.5D WFS driving function (2) under the validity of the SPA, i.e. the explicit solution is also valid locally for an arbitrary smooth reference curve within the Kirchhoff approximation. In the followings the above solutions are handled as being equivalent, allowing the spectral description of WFS driving functions in the spectral domain, when required.

Application of directive secondary sources

Traditional WFS and SDM theories assume secondary sources emitting omnidirectional spherical waves. First it is discussed, how the synthesized field inside the region of interest varies due to the application of directive SSD elements and theoretically how synthesis could be still optimized on the prescribed reference curve.

The field of a directive monopole can be written in the simple form

$$G_{\Theta}(\mathbf{x} - \mathbf{x}_0, \omega) = \Theta(\phi, \omega) G(\mathbf{x} - \mathbf{x}_0, \omega). \quad (4)$$

with the—potentially frequency dependent—secondary source directivity defined in the plane of synthesis by $\Theta(\phi, \omega) \in \mathbb{R}$. The polar angle $\phi = \arccos \left\langle \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} \cdot \mathbf{n}(\mathbf{x}_0) \right\rangle$ is the angle between the normal of the source element and the receiver point \mathbf{x} . This gives a fair approximation for the field of extended sources, with small spatial extension compared to the wavelength [11].

The synthesized field at a given receiver position is obtained as the convolution of the directive SSD elements and the driving function over the SSD

$$P_{\text{synth}}(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} D(\mathbf{x}_0) G_{\Theta}(\mathbf{x} - \mathbf{x}_0, \omega) d\mathbf{x}_0. \quad (5)$$

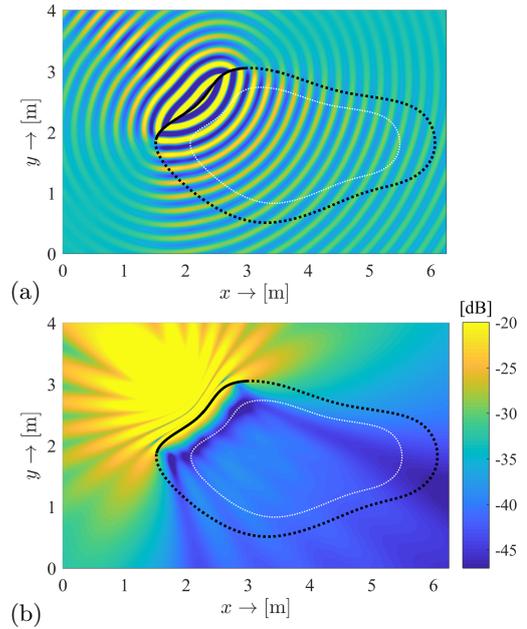


Figure 1: Synthesis of a harmonic virtual point source oscillating at $f_0 = 1.5$ kHz, located at $\mathbf{x}_s = [1.75, 3, 0]^T$ by applying an SSD with dipole directivity characteristics. The synthesis is referenced on a concentric reference curve with identical shape as the SSD. Figure (a) shows the real part of the synthesized field, while figure (b) shows the absolute discrepancy from the target sound field in a logarithmic scale. It is demonstrated that with the proposed directivity compensation approach amplitude correct synthesis may be achieved on the reference curve even with applying directive SSD elements.

According to the SPA the synthesized field is dominated by the stationary SSD element \mathbf{x}_0^* at which the phase gradient vanishes, i.e. where $\mathbf{k}^P(\mathbf{x}_0^*(\mathbf{x})) = \mathbf{k}^G(\mathbf{x} - \mathbf{x}_0^*(\mathbf{x}))$ holds. Here it was exploited that the directivity is real, not altering the secondary wavefront propagation direction.

Applying the SPA the amplitude factor of integral (5), i.e. the source directivity factor can be approximated around its stationary SSD element, resulting in the synthesized field

$$P_{\text{synth}}(\mathbf{x}, \omega) = \Theta(\phi_0(\mathbf{x}_0^*(\mathbf{x})), \omega) \underbrace{\int_{-\infty}^{\infty} D(\mathbf{x}_0) G(\mathbf{x} - \mathbf{x}_0, \omega) d\mathbf{x}_0}_{P_{\text{ideal}}(\mathbf{x}, \omega)} \quad (6)$$

where $\phi_0(\mathbf{x}_0^*(\mathbf{x})) = \arccos \langle \mathbf{k}^P(\mathbf{x}_0^*(\mathbf{x})) \cdot \mathbf{n}(\mathbf{x}_0^*(\mathbf{x})) \rangle$ is the stationary angle. This results states that the alteration of the synthesized field, compared to the result of ideal synthesis is given by the directivity function of the stationary SSD element into the direction of investigation, which is defined by the local propagation direction of the virtual wave field.

Restricting the investigation to the receiver curve in the ideal case—i.e. with omnidirectional SSD elements—the synthesized field would coincide with the virtual field within the validity of the SPA, with the alteration due to source directivity given by (6). Hence, over the reference curve the source directivity factor may be compensated

by applying the driving functions

$$D_{\text{WFS}}(x_0, \omega) = \sqrt{\frac{8\pi}{jk}} \sqrt{d_{\text{ref}}(\mathbf{x}_0)} \frac{jk_{\mathbf{n}}^P(\mathbf{x}_0)P(\mathbf{x}_0, \omega)}{\Theta(\phi_0(\mathbf{x}_0), \omega)}, \quad (7)$$

with $\phi_0(\mathbf{x}_0) = \arccos \langle \mathbf{k}^P(\mathbf{x}_{\text{ref}}(\mathbf{x}_0)) \cdot \mathbf{n}(\mathbf{x}_0) \rangle$. Note that the same conclusion may be drawn by directly substituting the transfer function of a directive source (4) into the spatial explicit driving function, which approach is inherently not restricted to monopole secondary sources. This result can be regarded as the generalization of the SSD directivity compensation introduced for linear SSDs and virtual point sources by [8].

The above compensation may be performed as long as the directivity function under discussion does not exhibit zeros as it is demonstrated in Figure 1. However, as it is revealed in the following section the application of directive secondary sources is particularly feasible in the aspect of reducing the effects of spatial aliasing due to discrete secondary source distribution.

Application of discrete secondary source distribution

So far synthesis, applying a continuous SSD was investigated. In practical applications the secondary loudspeakers are located at discrete positions. As a result behind the primary wavefront high-pass filtered echoes are radiated from each secondary source position—termed as spatial aliasing components—, as demonstrated in Figure 2 (a).

An advantage of the SDM is that it allows the analytical investigation of such aliasing phenomena. The discretization of the SSD is modeled as the sampling of the driving function with the sampling distance being the actual loudspeaker spacing Δx . In the wavenumber domain the sampled driving function reads the continuous driving function spectrum, repeating on multiples of the sampling wavenumber $2\pi/\Delta x$:

$$\tilde{D}_S(k_x, \omega) = \sum_{\eta=-\infty}^{\infty} \tilde{D} \left(k_x - \eta \frac{2\pi}{\Delta x}, \omega \right). \quad (8)$$

Spatial aliasing artifacts then can be described as the overlapping of the repeating spectrum, and as the reproduction of wavenumber components above the Nyquist-wavenumber $\pi/\Delta x$.

Bandlimiting the driving functions in the spatial domain: To avoid spectral overlapping appropriate filtering of the continuous driving function is required in order to bandlimit it to the Nyquist wavenumber. The stationary phase approximation allows the assignment of spatial coordinates to any given wavenumber component in the spectrum, identifying locations on the SSD from which aliasing components are radiated at a given temporal frequency.

According to [7] the spectrum of the driving function $\tilde{D}(k_x, \omega)$ at a given wavenumber component k_x is dominated by that position of the SSD, where

$$k_x^P(x_0) = \frac{\omega}{c} \hat{k}_x^P(x_0) = k_x \quad (9)$$

holds, with $\mathbf{k}(x_0)$ being the *normalized local wavenumber vector*: i.e. where the propagation direction of a horizontal spectral plane wave described by k_x coincides with the local propagation direction of the virtual field. Hence, that parts of the SSD will emit aliasing components, where locally

$$|k_x^P(x_0)| = \frac{\omega}{c} |\hat{k}_x^P(x_0)| \geq \frac{\pi}{\Delta x}. \quad (10)$$

Reformulating for an arbitrary shaped SSD the overlapping of the wavenumber spectra may be avoided by position-dependent temporal filtering with suppressing frequency components above the cutoff frequency

$$\omega \geq \frac{\pi}{\Delta x} \frac{c}{|\hat{k}_t^P(x_0)|}. \quad (11)$$

Here $\hat{k}_t^P(x_0)$ is the tangential component of the normalized wavenumber vector at a given SSD element. The above bandlimit criterion restricts temporally fullbandth synthesis to parts of the space for which the corresponding stationary SSD elements normal points to the local propagation direction of the virtual wavefront, as it can be seen in Figure 2 (b).

Bandlimiting the secondary source elements:

Even with ideal bandlimitation of driving functions, aliasing echoes are still present in the reproduced field due to the reproduction of wavenumber components above the Nyquist wavenumber, belonging to the adjacent mirror spectra. In the wavenumber domain the spectrum of the reproduced wave field with a linear SSD can be expressed as a spectral multiplication

$$\tilde{P}_{\text{synth}}(k_x, y, \omega) = \tilde{D}_S(k_x, \omega) \tilde{G}(k_x, y, \omega) \quad (12)$$

reflecting that aliasing could be entirely avoided by also bandlimiting the transfer function of the individual SSD elements—described by $\tilde{G}(k_x, y, \omega)$ —to the Nyquist wavenumber.

Assume the spatial Fourier transform of a directive point source, described by (4)

$$\tilde{G}_{\Theta}(k_x, y, \omega) = \int_{-\infty}^{\infty} \Theta(\phi(\mathbf{x}), \omega) G(\mathbf{x}, \omega) e^{jk_x x} dx. \quad (13)$$

Since it is assumed that the directivity function is real the stationary point for the Fourier integral is found, where $k_x^G(\mathbf{x}^*(k_x)) = k_x$ holds, and evaluation of the integral by using the SPA leads to

$$\tilde{G}_{\Theta}(k_x, y, \omega) \approx \Theta(\phi(\mathbf{x}^*(k_x)), \omega) \tilde{G}(k_x, y, \omega). \quad (14)$$

At a given wavenumber component $\Theta(\phi(\mathbf{x}^*(k_x)), \omega)$ describes the directivity of the SSD element into the direction, described by k_x . Since a given k_x corresponds to a propagation direction of an angle $\phi = \arcsin \frac{k_x}{k}$ measured from the y -axis (the normal of the SSD element in case of a linear array), therefore the above equation can be reformulated as

$$\tilde{G}_{\Theta}(k_x, y, \omega) \approx \Theta \left(\arcsin \frac{k_x}{k}, \omega \right) \tilde{G}(k_x, y, \omega). \quad (15)$$

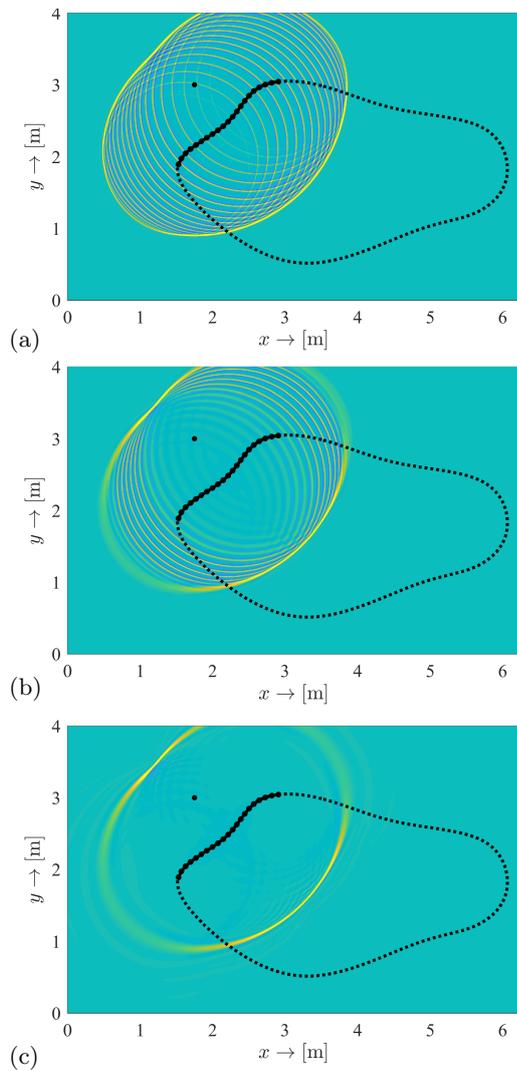


Figure 2: Synthesis of a wideband virtual point source with an excitation of a Gaussian impulse. The discrete SSD ($\Delta x \approx 10$ cm) is denoted by dashed black line, with black dots denoting the positions of the active SSD elements. Figure (a) shows fullband synthesis, with strong aliasing components behind the primary wavefront. Figure (b) presents the result of bandlimiting the driving functions temporally to (11). Although the alias echoes are minimized in the direction, where the stationary SSD elements normal coincides with the virtual field local propagation direction ($k_t \sim 0$), strong aliasing lateral wavefronts remain in the synthesized field. These waves are suppressed by applying ideal spatial low-pass filtering, as presented in figure (c).

Hence, antialiasing components may be suppressed in the reproduced field by applying an SSD consisting of directive SSD elements, for which ideally

$$\Theta(\phi, \omega) = 0, \quad \text{if } \sin \phi \geq \frac{\pi c}{\Delta \omega} \quad \text{or} \quad \omega \geq \frac{\pi c}{\Delta \sin \phi} \quad (16)$$

holds. Thus, in practical applications directive SSD elements are particularly beneficial in the aspect of suppressing aliasing components. Obviously, the above „ideally directed SSD elements” are not realizable, however the presented framework is useful for predicting the suppression factor in the wavenumber region, once the directivity of the applied SSD elements is known.

Conclusion

The contribution presented an analytical discussion of two well-studied non-idealities present in practical WFS application: the effect of directive SSD elements and spatial aliasing due to discretized SSD. It was shown that the artifacts, related to both phenomena can be efficiently described in the context of unified wave field synthesis theory, by applying the stationary phase approximation and the local wavenumber concept. Furthermore, approaches were presented in order to suppress the emerging frequency dependent amplitude errors and aliasing artifacts.

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