Topological design of fluid-structure interaction system using the mixed finite element formulation and boundary element method

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Introduction

Topology optimization has been already applied to fluid-structure interaction (FSI) to achieve a better performance. For bounded fluid domain, the coupling of structural finite element method (FEM) and acoustic finite element method is proved to be an efficient tool. However, the infinite or semi-infinite fluid domain increases the computational efforts of acoustic FEM rapidly. For such problems, the coupling of structural FEM and acoustic BEM, forming the FEM–BEM approach [1], has shown superiority. Optimizing FSI system, however, yields one extra issue, i.e., the variation of fluid–structure interface caused by the topology optimization. Some researchers only considered the material design of structure [2, 3], whereas the fluid–structure interface remains the same in the optimization process. Another approach is the mixed displacement–pressure ($\mathbf{u}/p$) FEM, where the structural and fluid domains are overlapped. It was firstly used by Yoon et. al [4] for the topology optimization of pressure load problems. However, they only analyzed the bounded domain by the coupling of structural FEM and acoustic FEM, which becomes unfavorable due to the infinite fluid domain as explained above.

In this work, we perform the topological design of structure surrounded by unbounded fluid domain by combining the mixed FEM and BEM based on our experiences of FEM–BEM [5, 6]. This coupled numerical model owns the benefits of mixed FEM and conventional FEM–BEM, which can handle both fluid–structure and fluid–fluid couplings. By using this analyzing model, the infinite analyzed domain is truncated, where the interior domain is analyzed by the mixed FEM. In this case, the computational costs of FEM are decreased obviously.

Mixed finite element discretization

Assuming an analyzing domain consisting of structural domain Ω_s and infinite fluid domain Ω_f. $\Gamma_{sf} := \Omega_s \cap \Omega_f$ is the fluid–structure interface. By choosing an appropriate virtual boundary Γ_{ie}, the infinite fluid domain Ω_f is truncated into a bounded one Ω_f^i in contact with Ω_s and an unbounded one Ω_f^e. Apparently, $\Gamma_{ie} = \Omega_f^i \cap \Omega_f^e$. Analyzing the mixed domain $\Omega_m = \Omega_s \cup \Omega_f^i$ by the mixed pressure–displacement (see, e.g., Chapter 10 of Reference [7]) yields the following linear system

$$\begin{bmatrix} \mathbf{K}_{uu} - \omega^2 \mathbf{M}_{uu} & -\mathbf{L}_{up} \\ -\mathbf{L}_{pu} & -\mathbf{K}_{pp} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}, \quad (1)$$

where $\mathbf{u}$ and $\mathbf{p}$ denote the structural displacement and pressure vectors, respectively. Matrices $\mathbf{K}_{uu}$, $\mathbf{K}_{pp}$, $\mathbf{M}_{uu}$ and $\mathbf{f}$ represent the stiffness matrices of displacement and pressure, mass matrix and external force vector. $\mathbf{L}_{up}$ and $\mathbf{L}_{pu}$ are coupling matrices, $\mathbf{L}_{up} = \mathbf{L}_{pu}^{\mathrm{T}}$. All the submatrices are sparse. Specially, $\mathbf{K}_{pp}$ is a block diagonal matrix since the pressure nodes are located inside elements. By employing the Schur component to eliminate the pressure unknowns, we have

$$\mathbf{K}\mathbf{u} = \left(\mathbf{K}_{uu} - \omega^2 \mathbf{M}_{uu} + \mathbf{L}_{up}\mathbf{K}_{pp}^{-1}\mathbf{L}_{pu}\right)\mathbf{u} = \mathbf{f}, \quad (2)$$

in which the matrix $\mathbf{K}$ is dense because the term $\mathbf{L}_{up}\mathbf{K}_{pp}^{-1}\mathbf{L}_{pu}$ is a dense matrix. Due to the dense pattern, the direct solution is computationally expensive, and the iterative solution also convergence slowly due to its ill-condition. In this case, we choose the sparse direct solver to solve equation (1) rather than (2).

Coupled formulations

After reconginizing the formulation of mixed FEM into the standard formulation, the coupling of mixed FEM and BEM becomes the same with the conventional FEM–BEM coupling. The fully coupled system is given by [1]

$$\begin{bmatrix} \mathbf{K} & -\mathbf{C}_{sf} \\ -\mathrm{i}\omega\mathbf{G}\boldsymbol{\Theta}^{-1}\mathbf{C}_{fs} & \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p}_f \end{bmatrix} = \begin{bmatrix} \mathbf{f}_s \\ \mathbf{p}_i \end{bmatrix}, \quad (3)$$

where $\mathbf{H}$ and $\mathbf{G}$ are the coefficient matrices of the BEM [8], $\mathbf{p}_f$ is the sound pressure on the boundary and $\mathbf{p}_i$ is the vector resulting from the incident wave. $\mathbf{C}_{sf}$ and $\mathbf{C}_{fs}$ are the coupling matrices, $\mathbf{C}_{sf} = \mathbf{C}_{sf}^{\mathrm{T}}$, $\boldsymbol{\Theta}$ is the boundary mass matrix, and $\mathbf{f}_s = \mathbf{f} - \mathbf{C}_{sf}\mathbf{p}_f$ is the structural load vector. Direct solution of equation (3) will be slow to converge due to its ill-conditioning. An alternative is to employ the Schur complement to eliminate the displacement unknowns and solve the pressure unknowns [1]. In the solution, the fast multipole method (FMM) [6] is applied to calculate the matrix-vector products of the BEM matrices. Then, we can compute the displacements $\mathbf{u}$ by a backward step.

Optimization model

The common optimization problem subjects to a given material volume fraction is formulated as follows:

$$\begin{cases} \min: & \Pi = \Pi(\mathbf{u}, \mathbf{p}_f) \\ \text{s.t.}: & \sum_{e=1}^{N_e} \mu_e v_e - f_v \sum_{e=1}^{N_e} v_e \leqslant 0, \\ : & 0 \leqslant \mu_{\min} \leqslant \mu_e \leqslant 1, \end{cases} \quad (4)$$

where an objective function Π is expressed as the function of sound pressures $\mathbf{u}$ and $\mathbf{p}_f$, μ_e and v_e denote the artificial density and volume of element e ($e = 1, 2, \ldots, N_e$) respectively, f_v denotes the volume fraction constraint, and $\mu_{\min}$ is the lower bound to avoid singular value in the calculation. In addition, the above optimization problem is also subject to the coupled governing equation (3). The method of moving asymptotes (MMA) [9] is employed to solve the optimization problem. Thereby, the sensitivity information is necessary.

Design sensitivity analysis

Following our previous work [10], the derivative of objective function with respect to e-th design variable can be formulated by the adjoint method as

$$\frac{\partial \Pi}{\partial \mu_e} = \Re\left(\boldsymbol{\lambda}_1^{\mathrm{T}} \frac{\partial \mathbf{K}_d}{\partial \mu_e}\mathbf{u} + z_3\right), \tag{5}$$

where $\boldsymbol{\lambda}_1^{\mathrm{T}}$ and $\boldsymbol{\lambda}_2^{\mathrm{T}}$ are the adjoint vectors resulting from solving the adjoint equation

$$\begin{cases} \mathbf{z}_1^{\mathrm{T}} + \boldsymbol{\lambda}_1^{\mathrm{T}}\mathbf{K} - \omega^2\rho\boldsymbol{\lambda}_2^{\mathrm{T}}\mathbf{G}\boldsymbol{\Theta}^{-1}\mathbf{C}_{fs} = 0, \\ \mathbf{z}_2^{\mathrm{T}} - \boldsymbol{\lambda}_1^{\mathrm{T}}\mathbf{C}_{sf} + \boldsymbol{\lambda}_2^{\mathrm{T}}\mathbf{H} = 0, \end{cases} \tag{6}$$

where $\mathbf{z}_1^{\mathrm{T}}$, $\mathbf{z}_1^{\mathrm{T}}$ and z_3 are obtained by applying the chain rule as

$$\frac{\partial \Pi}{\partial \mu_e} = \Re\left(\mathbf{z}_1^{\mathrm{T}} \frac{\partial \mathbf{u}}{\partial \mu_e} + \mathbf{z}_2^{\mathrm{T}} \frac{\partial \mathbf{p}_f}{\partial \mu_e} + z_3\right), \tag{7}$$

where z_3 does not contain the derivatives of state variables, i.e., $\partial \mathbf{u}/\partial \mu_e$ and $\partial \mathbf{p}_f/\partial \mu_e$. The adjoint method is therefore not free since it requires to solve the extra adjoint equation. It, however, only needs to be solved once because it does not contain derivative components. In addition, the solution of adjoint equations can also be accelerated by the FMM. The derivative of matrix $\mathbf{K}$ can be simply computed according to the RAMP interpolation [11] between solid and fluid.

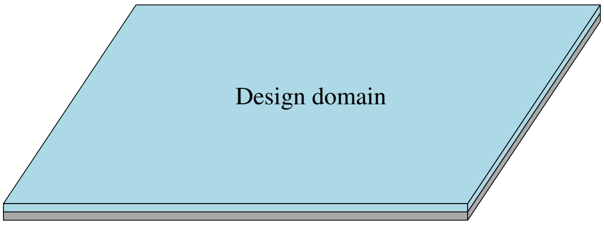
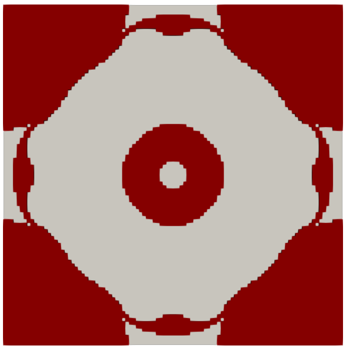


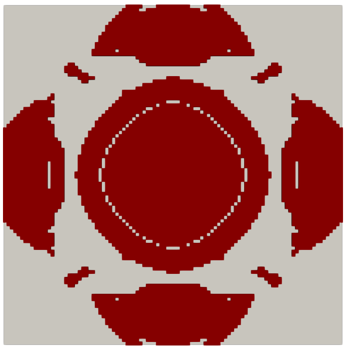
Figure 1: Plate example structure.

Plate example

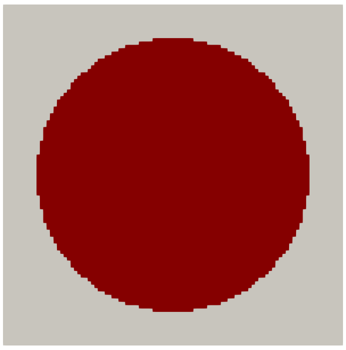
We consider a plate of dimensions 1 m × 1 m × 0.02 m immersed in water as shown in Figure 1. The plate is divided into two layers, where the bottom layer is the base structure and the top layer is the design domain. A harmonic load $F = 100\mathrm{e}^{-\mathrm{i}\omega t}$ is applied at the center point of the bottom plane. Steel is chosen as the elastic material with $E = 210$ GPa, $\nu = 0.3$ and $\rho_s = 7860$ kg/m^3. Water is the acoustic medium with $c_f = 1482$ m/s and $\rho_f = 1000$ Kg/m^3. Structural damping is ignored. The plate is discretized by 30603 vertices, 20000 mixed solid finite elements, and 20080 four-node constant boundary elements are used. Hence, the DOFs of the FEM and BEM are 111809 and 20800, respectively.



(a) $f_p = 50$ Hz.



(b) $f_p = 100$ Hz.



(c) $f_p = 150$ Hz.

Figure 2: Optimized designs for the plate example.

Regarding the optimization, the objective function is selected to be the objective function and the number of design variables is 10000 since the top layer is the design domain. We set material volume fraction limit as $f_v = 0.5$, and start the optimization from initial value being 0.6. Three frequencies, $f_p = 50$, 100 and 150 Hz are investigated, and the optimized distributions are illustrated in Figure 2. From the figure, we can see that three frequencies produce three different designs, which demonstrates the dependency on loading frequency.

In Table 1, we list the radiated sound power level with repescted to 1×10^{-12} W for the initial and optimized deigns. From the Table, It can be seen that significant decreases of sound power level are produced by the optimization. This demonstrates the validity of the proposed optimization method.

Table 1: Objective functions of the six designs.

	Initial design (dB)	Optimized design (dB)
50 Hz	65.66	61.47
100 Hz	70.81	64.84
150 Hz	81.15	76.99

Conclusions

In this paper, we combine the mixed displacement–pressure finite element method and the boundary element method for optimizing the FSI systems. By using this coupled approach, handling the variations of fluid–structure interface becomes straightforward. For efficiently calculating the objective function's gradients, an adjoint variable method (AVM) has been developed for the FSI systems with the aid of the fast multipole method. This reduces the computational effort. In the numerical tests, a plate example is used for verifying the used optimization method, and the optimization results demonstrate the proposed approach could decrease the sound radiation efficiently.

Acknowledgments

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