Sound Waves as Perturbations of Acoustic Spacetime

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Introduction

The analogy between acoustics in fluids and general relativity is established by W. G. Unruh [1], who used it for studying Hawking radiation indirectly, by means of better understood acoustic phenomena in transonic flows. Since then, the analogy has been further developed and promoted by M. Visser and C. Barceló in the so-called analogue models of general relativity [2] [3]. The authors showed that sound propagation in background flows can be described by the differential geometry of a curved spacetime. The analogy provided a new approach to complex gravitational phenomena with the aid of simpler non-relativistic (Newtonian) physics.

On the other hand, the relativistic approach has been used very sporadically for treatment of acoustic problems. The recent works of Gregory et al. [4] [5] are rare examples in this direction. Following Unruh and others, they treat sound propagation in uniform and non-uniform background flows by using geometric algebra in four-dimensional spacetime. Besides this, it should be mentioned that the Lorentz transformations, which are typical for special relativity, have been used for sound propagation in uniform mean flows [6] [7].

Although well established, the analogy appears to be limited only to four-dimensional geometrical interpretations of sound propagation. The reason are different governing equations in general relativity (Einstein field equations) and fluid dynamics [2] [5]. However, in this work we try to expand the analogy to the generation of sound in acoustic spacetime by hypothesizing that the sound waves are, analogously to the gravitational waves in general relativity, small perturbations of acoustic spacetime. For their description, we will use Einstein field equations as the governing equations with the stress-energy tensor as the source of the perturbations.

Sound waves in acoustic spacetime

Einstein field equations relate curvature of spacetime with its source, the second-order stress-energy tensor $\boldsymbol{T}$:

$$\boldsymbol{G} + \Lambda \boldsymbol{g} = \frac{kG}{c_0^4}\boldsymbol{T}, \tag{1}$$

where $\boldsymbol{G}$ is Einstein tensor, $\boldsymbol{g}$ is metric tensor, and Λ, k, G, and c_0 are constants, with the last one being speed of sound according to the acoustic spacetime analogy. Conservation of energy and momentum are expressed compactly by the equality:

$$G^{\alpha\beta}{}_{;\beta} = T^{\alpha\beta}{}_{;\beta} = 0, \tag{2}$$

with ; denoting covariant derivative in a curved four-dimensional spacetime ($\alpha, \beta = 0...3$) and Einstein summation convection supposed. Since we are not interested in steady (or slowly varying) solutions, we can adopt $\Lambda = 0$. In fact, we will assume that the only disturbance of otherwise flat spacetime is due to the waves. The metric tensor can then be written as a sum of the Minkowski metric, $\eta^{\alpha\beta} = \mathrm{diag}[-1, 1, 1, 1]$, and a weak perturbation component $h^{\alpha\beta}$:

$$g^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta}, \tag{3}$$

with $|h^{\alpha\beta}| \ll 1$.

Within the first-order approximation, it can be shown [8] that there always exists $\bar{h}^{\alpha\beta}$ ($|\bar{h}^{\alpha\beta}| \ll 1$) related to $h^{\alpha\beta}$, such that

$$\Box \bar{h}^{\alpha\beta} = -\frac{2kG}{c_0^4}T^{\alpha\beta}, \tag{4}$$

under the Lorenz gauge condition:

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0. \tag{5}$$

Here , is usual derivative (for example, $\phi_{,\alpha} = \partial\phi/\partial x^\alpha$) and $\Box = -(1/c_0^2)\partial^2/\partial t^2 + \nabla^2$ is d'Alembertian in flat spacetime with the coordinates $(x^1, x^2, x^3, x^4) = (c_0 t, x, y, z)$. Equations (4) are also called linearized Einstein field equations, which are governing equations for the linearized theory.

We will first exclude the source term and consider only wave propagation. Since the background spacetime is flat, eq. (4) becomes homogeneous wave equation which has a solution in the form of a plane wave, the real part of $\bar{h}^{\alpha\beta} = A^{\alpha\beta}e^{jk_\mu x^\mu}$. Components of the polarization tensor $A^{\alpha\beta}$ are complex in general. Since the metric tensor is symmetric, it has maximum 10 different components and 6 with the Lorenz gauge condition imposed. This number can be decreased further by choosing a suitable gauge.

In general relativity, transverse-traceless gauge is used for plane transverse gravitational waves. However, much more suitable for longitudinal acoustic waves in fluids appears to be Newtonian gauge (which is commonly used for deviations of the Newtonian gravitational potential). To show this, we first note that every free particle in spacetime obeys the geodesic equation. If the particle is moving non-relativistically (with velocity much smaller than c_0) in almost flat spacetime, the geodesic equation expresses its acceleration under the influence of a small metric perturbation $h_{\alpha\beta}$ as [8]:

$$\frac{d^2x^l}{dt^2} = -\frac{c_0^2}{2}\eta^{lk}(h_{k0,0} + h_{0k,0} - h_{00,k}), \tag{6}$$

where the Latin subscripts and superscripts refer to the spatial coordinates only, $k, l = 1...3$. In a Newtonian form, in which $|h_{0k}| \ll |h_{00}|$, motion of the particle due to the wave can be described with a single scalar h_{00}, as we expect from the acoustic waves in fluids:

$$\frac{d^2x^l}{dt^2} = \frac{c_0^2}{2} h_{00}^{;l}. \tag{7}$$

In the following, we will inspect how the component h_{00} is generated by different sources of sound.

Aeroacoustic quadrupole source

If all particles of a perfect fluid move with non-relativistic velocities $|\vec{v}| \ll c_0$ (such as in low Mach number subsonic flows), the approximation $\vec{U} \approx (1, \vec{v}/c_0)$ holds for the four-velocity $\vec{U}$. Components of the stress-energy tensor in the flat spacetime are then:

$$T^{\alpha\beta} = \begin{bmatrix} \rho c_0^2 & \rho c_0 v^1 & \rho c_0 v^2 & \rho c_0 v^3 \\ \rho c_0 v^1 & \rho v^1 v^1 + p & \rho v^1 v^2 & \rho v^1 v^3 \\ \rho c_0 v^2 & \rho v^1 v^2 & \rho v^2 v^2 + p & \rho v^2 v^3 \\ \rho c_0 v^3 & \rho v^1 v^3 & \rho v^2 v^3 & \rho v^3 v^3 + p \end{bmatrix}, \tag{8}$$

where ρ is prevailing matter density and $p \ll \rho c_0^2$ is pressure. We shall notice the similarity between its spatial part, T^{ij}, and Lighthill's tensor [9], which is a pure aeroacoustic source of sound in fluids. This is because the stress-energy tensor contains the conserved quantities, (see eq. (2)).

Next we follow Misner et al. [10] and consider a single isolated source of waves, far from which the spacetime is asymptotically flat towards the infinity (with the background metric $\eta^{\alpha\beta}$). Small metric perturbation is defined by eq. (3) and linearized eq. (4) holds under the condition in eq. (5). The stress-energy tensor can be formally split into effective stress-energy tensor $T^{\text{eff}}_{\alpha\beta}$ and a typically small component $t_{\alpha\beta}$ which is responsible for the radiation of waves: $T_{\alpha\beta} = T^{\text{eff}}_{\alpha\beta} + t_{\alpha\beta}$. Thus, we have:

$$\Box \bar{h}_{\alpha\beta} = -\frac{2kG}{c_0^4}(T^{\text{eff}}_{\alpha\beta} + t_{\alpha\beta}). \tag{9}$$

General solution for both ingoing ($\epsilon = -1$) and outgoing ($\epsilon = +1$) wave is:

$$\bar{h}_{\alpha\beta} = \frac{kG}{2\pi c_0^4} \int \frac{[T^{\text{eff}}_{\alpha\beta} + t_{\alpha\beta}]_{(t-\epsilon R/c_0)}}{R} d^3\vec{y}, \tag{10}$$

where the integral is over the entire space and $R = |x^i - y^i|$ is distance of the receiver location from the point y^i. The values of $T^{\text{eff}}_{\alpha\beta}$ and $t_{\alpha\beta}$ are to be evaluated at the retarded time $t - \epsilon R/c_0$.

Now we suppose that the source is compact, with the characteristic length scale $L \ll c_0/\omega$, where ω is characteristic angular frequency of the oscillations. Since $|\vec{v}| \sim L\omega$, this comes down to the slow-motion condition, $|\vec{v}| \ll c_0$. Furthermore, we consider far geometric field ($R \gg L$), so we can approximate:

$$\bar{h}_{\alpha\beta} = \frac{kG}{2\pi r c_0^4} \int [T^{\text{eff}}_{\alpha\beta} + t_{\alpha\beta}]_{(t-\epsilon r/c_0)} d^3\vec{y}, \tag{11}$$

where r is radial coordinate of a spherical coordinate system with the source in its origin. From the conservation laws in eq. (2), one can deduce the identity [10]:

$$\frac{1}{c_0^2}\frac{d^2}{dt^2} \int (T^{\text{eff}}_{00} + t_{00}) x_j x_k d^3\vec{x} = 2 \int (T^{\text{eff}}_{jk} + t_{jk}) d^3\vec{x}. \tag{12}$$

The integral on the left-hand side of the equation, which is the second moment of the mass distribution, is called quadrupole moment tensor of the mass distribution and often denoted with I_{jk}:

$$I_{jk} = \frac{1}{c_0^2} \int (T^{\text{eff}}_{00} + t_{00}) x_j x_k d^3\vec{x}. \tag{13}$$

The quantity which appears to be more convenient for description of wave generation is reduced quadrupole moment:

$$\rlap{--}I_{jk} = I_{jk} - \frac{1}{3}\delta_{jk} I_l{}^l, \tag{14}$$

where δ_{jk} is Kronecker delta and $I_l{}^l$ is the trace of I_{jk}. From equations (11)-(13):

$$\bar{h}_{jk} = \frac{kG}{4\pi r c_0^4} \frac{d^2}{dt^2} I_{jk}(t - \epsilon r/c_0). \tag{15}$$

In the rest of this section we limit ourselves to the derivation of Lighthill's scaling law for the acoustic power of the quadrupole source. Therefore, it is sufficient to estimate reaction of the source to the radiation, which can be done in the acoustic near field. Expanding $\bar{h}_{jk}$ in powers of r for $\omega r/c_0 \ll 1$ and leaving only the terms with ϵ (which correspond to the radiation[1]) gives in the Newtonian gauge (after replacing $\epsilon = 1$ for an outgoing wave) [10]:

$$h_{00}^{\text{react}} = -\frac{kG}{20\pi c_0^7} x^j x^k \frac{d^5}{dt^5} \rlap{--}I_{jk}(t) \tag{16}$$

to the lowest order, while $h_{0j}^{\text{react}} \sim (\omega L/c_0) h_{00}^{\text{react}}$ are of higher order. These components are reaction potentials (reaction to the radiation). Since they are in the Newtonian form, the metric component in eq. (16) represents the acoustically relevant perturbation of spacetime in the near field of the isentropic Lighthill's quadrupole $\rho\vec{v}\vec{v}$. The geodesic equation (7) gives the acceleration:

$$\frac{d^2x^l}{dt^2} = -\frac{kG}{40\pi c_0^5} \left(x^j x^k \frac{d^5}{dt^5} \rlap{--}I_{jk}(t) \right)^{,l}. \tag{17}$$

We can now derive Lighthill's scaling law for the radiated power. We suppose that T_{jk} scales as $\rho_0|\vec{v}|^2$ (an incompressible fluid). From equations (12)-(14), $|\rlap{--}I_{jk}| \sim \rho_0 L^5$ and from eq. (17), sound intensity scales as:

$$I \sim \rho_0 c_0 v_{\text{ac}}^2 \sim \frac{k^2 G^2 \rho_0^3 L^4}{c_0} \left(\frac{|\vec{v}|}{c_0} \right)^8, \tag{18}$$

[1] In the context of gravitation, the omitted terms which do not contribute to the radiation are corrections of the Newtonian potential causing the effects such as perihelion shift. In acoustic spacetime, they would give incompressible fluctuations which do not propagate into the far field.

where we also applied $\omega L \sim |\vec{v}|$. For $L \sim r$, Lighthill's 8[th] power law gives the scaling [7]: $I \sim \rho_0 c_0^3 (|\vec{v}|/c_0)^8$. Neglecting the multiplication constant, the two solutions become equal if we set

$$G = \frac{c_0^2 L}{2M} \sim \frac{c_0^2}{2\rho_0 L^2}, \tag{19}$$

where M is total mass of the source. In this way, we can identify L as the acoustic analogue to the Schwarzschild radius, which is characteristic length scale for sources of gravitational waves.

When $T_{00} \sim \rho c_0^2$ dominates sound generation (for example, in combustions), the power scales as $(\omega L)^4$, which is solely because the waves in question are longitudinal[2] (see also the following section). Dependence of the amplitude $\sim \omega^2$ appears in Lighthill's analogy as a consequence of the double differentiation of the source terms. However, the representation by means of acoustic spacetime appears to be more natural, since the full stress-energy tensor is used as the source, satisfying the conservation laws (eq. (2)). There is no need for splitting it into the source and acoustic propagation terms, or selecting an appropriate acoustic quantity for the analogy, since the waves are purely geometric perturbations of spacetime itself. We also do not contract the source tensors into scalars. The contraction is done by the choice of the Newtonian gauge for the resulting metric perturbation. Possible effects of the mean flow on sound propagation, such as convection and refraction, must be taken into account with appropriate background metric replacing the metric $\eta_{\alpha\beta}$ in eq. (3). This is discussed further in [4] and [5].

Pulsating sphere

Acoustic monopole has no counterpart in gravitation or electromagnetism (there are no monopole sources of gravitational or electromagnetic waves which originate from pulsating spherical objects). Therefore, it is worthwhile to study its appearance in the acoustic spacetime. We will solve eq. (9) in frequency domain, similarly as in [8], by supposing sinusoidal oscillations of $t_{\alpha\beta}$ in time with angular frequency ω: $t_{\alpha\beta} = S_{\alpha\beta} e^{-j\omega t}$. The source is compact, so $\omega L/c_0 \ll 1$, with L denoting radius of the sphere. Outgoing wave solution of eq. (9) in the far field has the form:

$$\bar{h}_{\alpha\beta} = \frac{A_{\alpha\beta}}{r} e^{-j\omega(t - r/c_0)}, \tag{20}$$

where r is distance from the source. Thereby, we neglect any terms of order r^{-2} (acoustic far field approximation). After removing the time dependence, we obtain Helmholtz type of the equation:

$$[(\omega/c_0)^2 + \nabla^2] \left(\frac{A_{\alpha\beta}}{r} e^{j\omega r/c_0} \right) = -\frac{2kG}{c_0^4} S_{\alpha\beta}. \tag{21}$$

[2]For comparison, electromagnetic and gravitational waves are transverse and described by a vector and a second-order tensor, respectively, which for a compact source gives higher orders of magnitude.

We can integrate the left-hand side of the equation over the source region, which gives the following terms:

$$\int_V \frac{\omega^2}{c_0^2} \frac{A_{\alpha\beta}}{r} e^{j\omega r/c_0} d^3\vec{y} = \frac{4\pi}{3} \left(\frac{\omega L}{c_0} \right)^2 A_{\alpha\beta} \tag{22}$$

and

$$\begin{aligned} \int_V \nabla^2 \left(\frac{A_{\alpha\beta}}{r} e^{j\omega r/c_0} \right) d^3\vec{y} = -4\pi A_{\alpha\beta} \\ + j4\pi \left(\frac{\omega L}{c_0} \right) A_{\alpha\beta}, \end{aligned} \tag{23}$$

with the approximation $e^{j\omega L/c_0} \approx 1$. The compressible acoustic waves must correspond to the term in eq. (22). Although stronger for compact sources (having lower order of $\omega L/c_0$), the terms in eq. (23) describe only transverse waves. In fact, the first term in it, which dominates for $\omega L/c_0 \ll 1$, contains the polarisation tensor of gravitational waves, for which the longitudinal component is omitted through gauging (to the transverse-traceless gauge). The acoustic waves are thus contained in the higher order gauge terms. The first term from eq. (23) and eq. (21) give the following expression for $A_{\alpha\beta}$ for the gravitational waves:

$$A_{\alpha\beta} = \frac{kG}{2\pi c_0^4} \int_V S_{\alpha\beta} d^3\vec{y}. \tag{24}$$

By comparing the term from eq. (22) with the first term from eq. (23), we can conclude that the acoustic waves are by the order $(\omega L/c_0)^2$ weaker and only a small fraction of $t_{\alpha\beta}$ generates the longitudinal waves:

$$A_{\alpha\beta} = -\frac{kG}{6\pi c_0^4} \left(\frac{\omega L}{c_0} \right)^2 \int_V S_{\alpha\beta} d^3\vec{y} \tag{25}$$

and from eq. (20):

$$\bar{h}_{\alpha\beta} = -\frac{kG}{6\pi c_0^4} \left(\frac{\omega L}{c_0} \right)^2 \frac{e^{j\omega r/c_0}}{r} \int_V t_{\alpha\beta} d^3\vec{y}. \tag{26}$$

This represents the acoustic perturbation of spacetime.

In the case of a pulsating sphere, the only non-zero component is $\bar{h}_{00}$. We can also express

$$\begin{aligned} \int_V t_{00} d^3\vec{y} &= \rho_0 c_0^2 \frac{4}{3} (L + \bar{l} e^{-j\omega t})^3 \pi \\ &\approx 4\rho_0 c_0^2 L^2 \pi \bar{l} e^{-j\omega t}, \end{aligned} \tag{27}$$

where $\bar{l} \ll L$ is amplitude of the oscillations around L, as well as $G = 3c_0^2/(8\rho_0 L^2 \pi)$ from eq. (19). Thus we obtain[3]:

$$\bar{h}_{00} = -\frac{k}{4\pi} \left(\frac{\omega L}{c_0} \right)^2 \bar{l} \frac{e^{-j\omega(t - r/c_0)}}{r}. \tag{28}$$

Since the components $\bar{h}_{0j}$ are zero, the metric is in the Newtonian form and from eq. (7) with r replacing l:

$$v_{\mathrm{ac}}^r = \frac{k}{8\pi} c_0 \left(\frac{\omega L}{c_0} \right)^2 \bar{l} \frac{e^{-j\omega(t - r/c_0)}}{r}. \tag{29}$$

[3]This agrees with the Schwarzschild metric which is the only spherically symmetric and asymptotically flat (for $r \to \infty$) solution of Einstein field equations in vacuum[10].

We again neglected the component $\sim 1/r^2$ in the acoustic far field. The metric does not produce any transverse waves and is indeed purely acoustical.

This matches the classical solution for a compact pulsating sphere [11] if we adopt $k = -8\pi$. In the theory of gravitation $k = 8\pi$. The opposite sign followed from different signs of the term in eq. (22) and the lowest order term in eq. (23). As a consequence, mass acts as an attracting source of gravity and (its unsteady component) as a repelling source in acoustics.

Oscillating sphere

As en example of acoustic dipole source, we consider a compact sphere which oscillates with velocity $\vec{v} \sim e^{-j\omega t}$ and magnitude $|\vec{v}|$. The derivation is the same as in the previous section, except that we replace the integral of $t_{\alpha\beta}$ in eq. (27) with:

$$\int_V t_{0j} d^3\vec{y} = \rho_0 c_0 L^3 \pi v_j. \tag{30}$$

The translating sphere is acting on the layer of the surrounding fluid with the scattering cross-sectional area of the sphere, $L^2\pi$. Equation (26) gives the only non-zero components of the metric perturbation:

$$\bar{h}_{0j} = -\frac{kG\rho_0 L^3}{6c_0^3}\left(\frac{\omega L}{c_0}\right)^2 \frac{e^{j\omega r/c_0}}{r} v_j. \tag{31}$$

In order to obtain a Newtonian gauge, we transform the coordinates $x^\alpha \to x^\alpha + \xi^\alpha$ with

$$\xi_0 = -\frac{kG\rho_0 L^3}{6c_0^3}\left(\frac{\omega L}{c_0}\right)^2 \frac{e^{j\omega r/c_0}}{r} v_j x^j \text{ and } \xi_j = 0 \tag{32}$$

In this gauge we calculate [8]:

$$\begin{aligned} h_{00} &= \bar{h}_{00} - \xi_{0,0} - \xi_{0,0} = -\frac{2}{c_0}\frac{\partial \xi_0}{\partial t} \\ &= -j\frac{k}{8\pi c_0}\left(\frac{\omega L}{c_0}\right)^3 \frac{e^{j\omega r/c_0}}{r} v_j x^j \end{aligned} \tag{33}$$

and $h_{0j}, h_{jk} = 0$ to the lowest order of $\omega L/c_0$ and $1/r$. We also replaced $G = 3c_0^2/(8\rho_0 L^2\pi)$. Now we can use eq. (7) and obtain the radial which is component proportional to $1/r$:

$$v_{\text{ac}}^r = \frac{kc_0}{16\omega\pi}\left(\frac{\omega L}{c_0}\right)^3 |\vec{v}|\cos(\theta)\frac{e^{-j\omega(t-r/c_0)}}{r}, \tag{34}$$

where we also expressed the radial component of $(v_j x^j)^{,r}$ as $v\cos(\theta)e^{-j\omega t}$ and θ is angle between $\vec{v}$ and the position vector $\vec{r}$. This matches the classical solution [11] for $k = -8\pi$, which also confirms the value of k from the previous section.

Conclusion

In this work we showed that Einstein field equations can be used as governing equations for acoustic problems. Sound waves in fluids were treated as longitudinal perturbations of acoustic spacetime. Aeroacoustic sound generation by the stress-energy tensor was also found to reproduce Lighthill's 8$^{\text{th}}$ power law. The solutions for archetypal sources, pulsating and oscillating sphere, match the classical results. Although we considered only the linearized theory, the fact that eq. (2) expresses the conservation laws of fluid dynamics suggests that general Einstein field equations from eq. (1) can also be adopted, with $k = -8\pi$ and G as in eq. (19). Nevertheless, possible applications for non-linear acoustics and high Mach number subsonic flows should be further investigated.

The linearized formulation for small perturbations of otherwise flat spacetime, eq. (4), holds under the Lorenz gauge condition in eq. (5). Particle motion due to a weak acoustic wave is then determined by the component h_{00} of the metric perturbation in the Newtonian gauge, according to eq. (7). A natural extension of the work would be to include the effects of boundaries within the considered relativistic framework, possibly with the aid of generalized functions, following Ffowcs Williams and Hawkings approach.

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