Changing perspectives in aeroacoustics

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Introduction

The engineering field of aeroacoustics deals with the numerical and experimental investigation of sound propagation and generation in moving fluids, especially air or water. For this purpose, suitable models must exist which represent the physical processes with suitable mathematical expressions.
The main ground-breaking ideas go back to Lighthill and Newman [8], and are known as acoustic analogies, see [7, 13] and references within. But there were also critical opinions on this approach, see Doak [2], who published a fabulous and easily understandable overview of the topic.
Alternative approaches consist of separating the physical quantities into mean values and fluctuations, whereby the fluctuations are generally regarded as much smaller than the mean values. A numerous number of publications are available on the so-called *Linearized Navier-Stokes Eequation* (LNSE), the *Linearized Euler Equation* (LEE), or the *Acoustic Perturbation Equation* (APE). For the latter see Ewert and Schröder [3]. All these approaches follow the *Eulerian* framework of continuum mechanics, which is typically used for investigating fluid dynamics. In contrast, the *Lagrange* frame is used less frequently. However, both frames possess advantages and consequently an *Arbitrary Lagrangian Eulerian* (ALE) framework was developed, see Belytschko et al. [1]. With this class of framework applied for aeroacoustics, an alternative approach known as *Galbrun* equation has been developed and studied, see [4–6, 9, 10, 12], in which the perturbations are defined in terms of a Lagrangian displacement.
Following this approach, possibles ways of deriving Galbrun's equation together with possible sources are discussed in this work.

Concept of ALE formulation

The basic concept of the ALE formulation is briefly discussed in this section, as this will be a preparation for the application to the conservation equations. Consider an Euclidean space in R^3 with a Cartesian coordinate system with te base $\boldsymbol{e}_i$ with $i = 1, 2, 3$. Two classical frameworks of continuum mechanics exist, namely the Lagrangian and the Eulerian frame. In the Lagrangian frame, the observer is fixed to the material or the continuum, whereas in the Eulerian frame the observer is fixed in space and sees the continuum pass by as it changes its configuration over time. In the ALE formulation another domain is introduced, which serves as a reference. As this reference can be chosen arbitrarily, the naming of the frame is plausible. For deeper insight, the reader is referred to the literature, see Belytschko et al. [1].
Figure 1 presents the basic configurations for the following relations. The main concept consists in the definition of a mixed representation with respect to the reference coordinates $\boldsymbol{x}$. First, consider a physical quantity Φ in an Eulerian frame. As mentioned in the introduction, separation in mean and perturbation yields

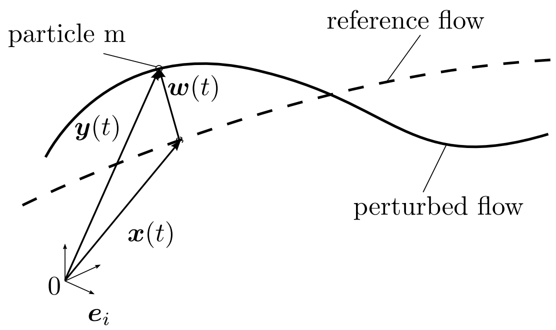

Figure 1: Basic configuration for describing aeroacoustics

$$\Phi(y_l, t) = \Phi_0(y_l) + \Phi'(y_l, t), \tag{1}$$

where the Eulerian perturbation is expressed with an Φ'. In the Lagrangian frames this separation would be with respect to an assigned material particle. However, in the mixed frame the separation of quantities can be formulated as

$$\Phi(y_l, t) = \Phi_0(x_l) + \tilde{\Phi}(x_l, t), \tag{2}$$

where $\tilde{\Phi}$ denotes the Lagrangian perturbation of an Eulerian quantity with respect to an arbitrary reference position $\boldsymbol{x}$. By utilizing equation (1) and (2) and considering small deformations, the usual Eulerian perturbation can be computed from the Lagrangian up to first order as

$$\Phi'(y_l, t) = \tilde{\Phi}(x_l, t) - w_j \left(\Phi_0(x_l, t)\right)_{,j}. \tag{3}$$

Note that for quantities followed by a comma and an index, the spatial derivative applies. Furthermore, Einstein's summation convention shall be used for repeating indices. From these definitions, a number of general expressions arise which are stated subsequently. First, the Lagrangian displacement is defined according to Figure 1 as

$$w_i(t) = y_i(t) - x_i(t). \quad (4)$$

Second, from basic principals of continuum mechanics the deformation gradient $\boldsymbol{F}$ and the Jacobian J read

$$\mathrm{d}y_i = F_{ij}\mathrm{d}x_j \quad \text{with} \quad F_{ij} = \delta_{ij} + w_{i,j}, \quad (5)$$

$$J = \det \boldsymbol{F} \quad \text{or} \quad J = \frac{1}{6} e_{lmn} e_{pqr} F_{lp} F_{mq} F_{nr}, \quad (6)$$

where e_{lmn} denotes the Levi-Civita symbol. Furthermore, the inverse of $\boldsymbol{F}$ reads

$$G_{ij} = F_{ij}^{-1} = \frac{1}{J} T_{ji} \quad \text{with} \quad (7)$$

$$T_{ij} = (1 + w_{l,l})\delta_{ij} - w_{j,i} + N_{ij} \quad \text{and} \quad (8)$$

$$N_{ij} = \frac{1}{2}\left(w_{l,l}^2 - w_{m,n} w_{n,m}\right)\delta_{ij} - w_{l,l} w_{j,i} + w_{j,l} w_{l,i}. \quad (9)$$

The equations stated above are particularly important for deriving transformation rules of the gradient and divergence operators with respect to the reference frame. These rules are

$$P_{,j}(y_l,t) = \Big(P_0(x_l,t) + \tilde{P}(x_l,t)\Big)_{,i} G_{ij}(x_l,t), \quad (10)$$

$$P_{j,j}(y_l,t) = \frac{1}{J(x_l,t)}\left[\Big(P_{0i}(x_l,t) + \tilde{P}_i(x_l,t)\Big) T_{ij}(x_l,t)\right]_{,j}, \quad (11)$$

where $\boldsymbol{P}$ represents a tensor of arbitrary but sufficient order. For deeper insight, the reader is referred to Minotti et al. [10].

All necessary information has been presented to derive Galbrun's equation.

Application to fluid dynamics

In order to apply the presented relations, the general equations of fluid dynamics must be recalled. In the conventional Eulerian frame, these equations read

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho v_{l,l} \quad (12)$$

$$\rho\frac{\mathrm{D}v_j}{\mathrm{D}t} = \sigma_{ji,i} + f_j \quad (13)$$

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t} = \sigma_{kl} v_{k,l} - q_{j,j} + \dot{\vartheta}. \quad (14)$$

and are known as balance of mass, momentum and energy. Here, ρ represents the fluid density, $\boldsymbol{v}$ the velocity, $\boldsymbol{\sigma}$ the Cauchy stress tensor, $\boldsymbol{f}$ body forces, e the specific inner energy, $\boldsymbol{q}$ the heat convection, and finally $\dot{\vartheta}$ additional heat sources. The equilibrium of angular momentum is implicitly satisfied by the symmetry of $\boldsymbol{\sigma}$, which is defined as

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}, \quad (15)$$

where p is the pressure and τ the viscous stress tensor. Using the well-known Gibb's theory, equation (14) can be converted to an entropy equation, which yields

$$\rho\frac{\mathrm{D}s}{\mathrm{D}t} = \frac{1}{T}\left[\tau_{kl} v_{k,l} - q_{j,j} + \dot{\vartheta}\right], \quad (16)$$

where T denotes the temperature. Last but not least, $\frac{\mathrm{D}()}{\mathrm{D}t}$ denotes the material time derivative.

For simplification, the following assumptions apply. First, heat sources and heat conduction are neglected. Second, only small perturbations are considered, which, after following the outlines of Minotti et al. [10], yields

$$J = 1 + w_{l,l}, \quad (17)$$

$$T_{ij} = \delta_{ij} - w_{j,i} + w_{l,l}\delta_{ij}, \quad (18)$$

$$\tilde{\rho} = -\rho_0 w_{l,l}. \quad (19)$$

As a next step, all presented relations are used and inserted into the balance equation of momentum, cf. equation (13), while considering the mixed representation of all quantities, cf. equation (2). In addition, the Lagrangian velocity perturbation is defined as the material time derivative of the Lagrangian displacement as $\tilde{\boldsymbol{v}} = \frac{\mathrm{D}\boldsymbol{w}}{\mathrm{D}t}$, where the material time derivative reads $\frac{\mathrm{D}()}{\mathrm{D}t} = \frac{\partial()}{\partial t} + v_{0k}()_{,k}$.

After some cumbersome rearrangements, the following mathematical expression can be stated.

$$\mathcal{G}\{w\} + \mathcal{G}_{\tau f}\{w\} + \mathcal{G}_{NL}\{w\} = \mathcal{S}_0 + \mathcal{S}_1, \quad (20)$$

where on the left-hand side of equation (20) all components are located that can be related to wave propagation and on the right-hand side, possible sources appear. They read

$$\mathcal{G}\{w\} = \rho_0 \frac{\mathrm{D}^2 w_k}{\mathrm{D}t^2} + p_{0,k} w_{q,q} - p_{0,l} w_{l,k} + \tilde{p}_{,k}, \quad (21)$$

$$\mathcal{G}_{\tau f}\{w\} = \tau_{0ki,j} w_{j,i} + \tau_{0ki} w_{j,ij} - \tau_{0kj,j} w_{l,l} - \tau_{0kj} w_{l,lj} - f_{0k} w_{l,l}, \quad (22)$$

$$\mathcal{G}_{NL}\{w\} = -\tilde{p}_{,j} w_{j,k} + \tilde{\tau}_{ki,j} w_{j,i} + \tilde{p}_{,k} w_{l,l} - \tilde{\tau}_{kj,j} w_{l,l} - \tilde{\tau}_{kj} w_{l,lj} - \tilde{f}_k w_{l,l}, \quad (23)$$

$$\mathcal{S}_0 = -\rho_0 \frac{\mathrm{D}v_{0k}}{\mathrm{D}t} - p_{0,k} + \tau_{0kl,l} + f_{0k}, \quad (24)$$

$$\mathcal{S}_1 = \tilde{\tau}_{kj,j} + \tilde{f}_k. \quad (25)$$

In detail, $\mathcal{G}\{w\}$ denotes the usual Galbrun operator, $\mathcal{G}_{\tau f}\{w\}$ contains wave propagation effects due to viscosity and body forces with respect to the reference flow Φ_0. Furthermore, $\mathcal{G}_{NL}\{w\}$ retains all nonlinear components arising from the mathematical derivation and can be set to zero, since only small perturbations are considered. Note that with regard to acoustic analogies, the nonlinear terms $\mathcal{G}_{NL}\{w\}$ can be shifted to right hand side. In this case, the nonlinear terms can be assumed as pseudo sources with respect to the linear propagation terms on the left. Such source terms could be calculated from a precise CFD simulation within the source region as part of a hybrid flow/acoustic analysis [13].

However, on the source side of equation (20), two main source contributions can be identified, where $\mathcal{S}_0$ contains all possible source components of the reference flow and $\mathcal{S}_1$ presents sources due to fluctuations within the viscous stress tensor as well as body force fluctuations. Therefore, equation (20) can be written as

$$\mathcal{M}\{w\} = \mathcal{G}\{w\} + \mathcal{G}_{\tau f}\{w\} = \mathcal{S}_0 + \mathcal{S}_1. \quad (26)$$

Since up to this point, nothing has been declared neither with respect to the constitutive equations nor to the reference flow, equation (26) can be seen as a generalized form of Galbrun's equation for small perturbations.
A similar approach can be applied to the entropy balance equation, cf. equation (16), which yields

$$\rho_0 \frac{\mathrm{D}\tilde{s}}{\mathrm{D}t} = \mathcal{E}_0 + \mathcal{E}_1, \tag{27}$$

where

$$\mathcal{E}_0 = -\rho_0 \frac{\mathrm{D}s_0}{\mathrm{D}t} + \frac{1}{T_0}\tau_{0kl}v_{0k,l}, \tag{28}$$

$$\mathcal{E}_1 = \frac{1}{T_0}\left(\tau_{0kl}\tilde{v}_{k,l} - \tau_{0kl}v_{0k,i}w_{i,l} + \tilde{\tau}_{kl}v_{0k,l}\right). \tag{29}$$

Note for this case that the temperature perturbation $\tilde{T}$ has been set to zero and again nonlinear terms have been dropped.

The perspective towards sound

To close the system of equations, suitable constitutive relations must be declared. Usually, when describing sound propagation as a fluctuation of the ambient pressure, it is a valid assumption that heat conduction can be neglected. This result stems from the fact that for low frequencies heat production due to conduction is weaker than for higher frequencies. Here, the term *low frequencies* includes the audible range, see Pierce [11]. Furthermore, an ideal gas is considered and the constitutive relation can be assembled as

$$\tilde{p} = c^2\tilde{\rho} + c^2\left(\frac{\rho_0}{c_p}\right)\tilde{s}. \tag{30}$$

It is assumed that the entropy perturbations for the thermodynamic process of wave propagation are zero and therefore $\tilde{s} = 0$. In the case of small perturbations, the pressure fluctuations are simply

$$\tilde{p} = -c_0^2\rho_0 w_{l,l}, \tag{31}$$

where equation (19) has been used. The final set of equations consists of (19), (26), together with (21), (22), (24), and (25) as well as (27) for which $\tilde{s} = 0$ and (28) together with (29), and last but not least (31).
It is now up to the specific case for which sources or propagation effects can be neglected in order to simplify the governing equations.

Discussion of sources

In this section, the sources of the generalized Galbrun's equation as well as the entropy equation are discussed. By taking a closer look at equation (24), one identifies that the reference quantities Φ_0 reassemble the momentum equation (13). So far, the reference flow is sill arbitrary. In the case that the reference quantities fulfill equation (13), the reference flow will not excite any wave propagation. In the case where the flow is stationary and any viscous effects as well as body forces are neglected, the only source of sound is the time-varying gradient of the reference pressure $-p_{0,k}$, which is in agreement with the results of Gabard et al. [4]. Note that likewise the generalized Galbrun's equation reduces to the usual form, since $\mathcal{G}_{\tau f}\{w\} = 0$. It can be seen that, depending on the restrictions towards the reference flow, sources of Galbrun's equation can easily be constructed.
Taking a closer look at the entropy equation (27) with its potential sources (28), and (29), while recalling that $\tilde{s} = 0$, one can see that

$$\rho_0 \frac{\mathrm{D}s_0}{\mathrm{D}t} = \frac{1}{T_0}(\tau_{0kl}v_{0k,l} + \tau_{0kl}\tilde{v}_{k,l} - \\ - \tau_{0kl}v_{0k,i}w_{i,l} + \tilde{\tau}_{kl}v_{0k,l}) \tag{32}$$

remains. Equation (32) can be seen as a physical restriction with respect to the reference flow.

Derivation from energy principles

As a final remark, the derivation of Galbrun's equation can be conducted based on energy principles, which are briefly discussed as follows.
All external sources of energy can be formulated as

$$U = F_k\frac{\partial w_k}{\partial t} + \tilde{p}\frac{\partial B}{\partial t}. \tag{33}$$

Here, F_k represents possible sources of

$$\rho_0\frac{\mathrm{D}^2 w_i}{\mathrm{D}t^2} + \tilde{p}_{,i} + p_{0\,,i}w_{l,l} - \frac{1}{\rho_0 c_0^2}p_{0\,,i}\left(\tilde{p} + w_k p_{0\,,k}\right) = F_i. \tag{34}$$

Furthermore, B represents external sources of

$$w_{l,l} + \frac{1}{\rho_0 c_0^2}p_{0\,,i}\left(\tilde{p} + w_k p_{0\,,k}\right) = B. \tag{35}$$

Note that by utilizing equations (34) and (35), Galbrun's equation can be recalled. After some manipulations, the following expression can be derived

$$\frac{\partial E}{\partial t} + I_{k,k} = U, \tag{36}$$

where

$$E = \frac{1}{2\rho_0 c_0^2}\left(\tilde{p}^2 - (w_k p_{0,k})^2\right) + \rho_0\frac{\partial w_k}{\partial t}\frac{\mathrm{D}w_k}{\mathrm{D}t} - \\ - \frac{\rho_0}{2}\frac{\mathrm{D}w_k}{\mathrm{D}t}\frac{\mathrm{D}w_k}{\mathrm{D}t} + \frac{w_k}{2}w_{k,l}p_{0,l} \tag{37}$$

and

$$I_k = \tilde{p}\frac{\partial w_k}{\partial t} + \rho_0 v_{0k}\left(\frac{\partial w_k}{\partial t}\frac{\mathrm{D}w_k}{\mathrm{D}t}\right). \tag{38}$$

For the cases in which a gravity potential needs to be taken into account, one can refer to Godin [6].

Conclusion

In this work, after presenting a general introduction to field and the main concept of an ALE framework, the generalized concept was applied to fluid dynamics

in order to arrive at a generalized form of Galbrun's equation. Note that only small perturbations where considered. This form of Galbrun's equation allows one to retain possible sources depending on the assumption that apply to the reference flow. In addition a possible way of deriving Galbrun's equation from energy principals has been recalled.
With the presented outlines it is possible to easily identify possible sources of Galbrun's equation. Furthermore, the concept offers a possible way for extending the theory towards the analysis of problems for which thermo-viscous effects play an important role.

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