Lorentz Invariance and Lagrangian Formulations of Acoustics in Fluids

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Introduction

Physical formulation of acoustics in fluids is commonly acquired from the Newtonian equations of fluid dynamics and their simplifications. The alternative approach based on Lagrangian functions and the variational principle has been used sporadically, even though it is central for most of the modern field theories, such as electromagnetism, gravity, and quantum field theory [1, 2]. These theories are typically formulated with Lorentz-invariant Lagrangians defined in the four-dimensional spacetime. In the rest of this work it will be demonstrated that acoustic field is no exception in the theory based on analogous acoustic spacetime (with the speed of light replaced by the speed of sound) [3, 4].

In Ref. [5] it is discussed that sound waves produced by a quadrupole source (turbulent low Mach number flow) can be associated with weak perturbations of the background acoustic spacetime, which are described by the linearized Einstein field equations. This consideration will be extended here to the analogy with electromagnetism in the case of dipole radiation. In the following sections we inspect several Lagrangian formulations which can capture different acoustic phenomena in fluids. In particular, we show that essentially incompressible near acoustic field corresponds to adding mass to the massless gauge field associated with acoustics through the process of spontaneous symmetry breaking. Large but spatially confined mass violating the Lorentz invariance is used to represent rigid boundaries explicitly in the equation of motion, while the sources are modelled as Noether currents, which follow from the symmetries of Lagrangians describing the non-acoustical field of fluid particles. Finally, the aeroacoustic analogy of Ffowcs Williams and Hawkings [6] is derived in the acoustic spacetime.

Lagrangian and equation of motion

For a given Lagrangian (density) $\mathcal{L}$, action is defined as its integral over the entire spacetime:

$$S = \int d^4x \mathcal{L}(x), \tag{1}$$

where x is a point in spacetime defined by four coordinates $x^\mu = [c_0 t, \vec{x}]$ ($\mu = 0...3$) and here we suppose that c_0 is the reference speed of sound for acoustic problems. The equation of motion (for example, wave equation) is derived by the principle of least action under variation of the field. For example, for a scalar field ϕ Lagrangian is a functional only of ϕ and its first derivatives, $\phi_{,\mu}$. Here and in the following comma is used for the derivative with respect to the coordinate which follows it, that is, $\phi_{,\mu} = \partial\phi/\partial x^\mu$. Minimizing the action leads to the Euler-Lagrange equation [1]

$$\frac{\partial \mathcal{L}}{\partial \phi} - \left(\frac{\partial \mathcal{L}}{\partial(\phi_{,\mu})}\right)_{,\mu} = 0. \tag{2}$$

The most general Lorentz-invariant Lagrangian of a real scalar field is

$$\mathcal{L} = \frac{1}{2}\phi^{,\mu}\phi_{,\mu} + \frac{1}{2}\left(\frac{mc_0}{\hbar}\right)^2 \phi^2, \tag{3}$$

where m is a generic mass, $\hbar$ is reduced Planck constant, and the normalization with 1/2 is conventional. It is invariant under the discrete symmetry $\phi \to -\phi$. Inserting eq. (3) into eq. (2) gives the Klein-Gordon equation in the flat Minkowski spacetime:

$$-\left(\phi^{,\mu}\right)_{,\mu} + \left(\frac{mc_0}{\hbar}\right)^2 \phi = -\Box\phi + \left(\frac{mc_0}{\hbar}\right)^2 \phi = 0. \tag{4}$$

For $m = 0$, the equation is the homogeneous scalar wave equation. Hence, acoustic field is well described with the massless real scalar field Lagrangian.

Mass in acoustic near field

Non-acoustic massive ϕ is still of great interest. The Planck–Einstein relation reads $E = \hbar\omega$, where E is energy (not to be confused with acoustic energy) and ω is angular frequency. On the other hand, for a non-relativistic particle, $E = mc_0^2$. For the purpose of the discussion here, these two equations can be taken as the definition of $\hbar$: $\hbar = mc_0^2/\omega$. Equation (4) in frequency domain becomes

$$\left(\frac{\omega}{c_0}\right)^2 \phi + \nabla^2\phi - \left(\frac{mc_0}{\hbar}\right)^2 \phi = 0 \tag{5}$$

and after inserting $\hbar$ we obtain Laplace's equation $\nabla^2\phi = 0$, which is not Lorentz invariant and belongs to the realm of non-relativistic (Newtonian) field theory. In acoustics it describes incompressible (density $\rho =$ const. and $c_0 \to \infty$) fluctuations in the acoustic near field [7], at frequencies ω and distances l which satisfy $\omega l/c_0 \ll 1$. As will be shown later, mass is added in the acoustic near field, when the Lagrangian symmetry is spontaneously broken. The length scale L at which this happens is roughly given by the condition

$$\frac{\omega L}{c_0} = \frac{mc_0 L}{\hbar} = 1 \tag{6}$$

and it depends on frequency.

Mass as a thin homogeneous wall

When confined in space, the mass term can also be used for introducing effects of solid bodies and boundaries. In general they violate the Lorentz invariance. As an example we can consider a thin homogeneous wall. Supposing a one-dimensional space along the axis x_1 and replacing m^2 with $m^2 C\delta(x_1)$, where C is a constant, we write down the equation

$$\left(\frac{\omega}{c_0}\right)^2 \phi + \frac{d^2\phi}{dx_1^2} - \left(\frac{mc_0}{\hbar}\right)^2 C\delta(x_1)\phi = 0. \quad (7)$$

The general solution for $x_1 < 0$ is $\phi(x_1 < 0) = Ie^{jkx_1} - IRe^{-jkx_1}$ and for $x_1 > 0$ is $\phi(x_1 > 0) = ITe^{jkx_1}$, where $k = \omega/c_0$ is wave number, I is the complex amplitude of the incident plane wave and R and T are usual reflection and transmission coefficient, respectively. We assume zero energy losses, absence of an incoming wave from $x_1 = +\infty$, and we associate ϕ with velocity, which leads to the minus sign in the solution for $\phi(x_1 < 0)$. The two solutions can be related by integrating eq. (7) around $x_1 = 0$ [8], which gives

$$jk(IT - I - IR) - \left(\frac{mc_0}{\hbar}\right)^2 C(I - IR) = 0. \quad (8)$$

Continuity of the solution at $x_1 = 0$ forces $I - IR = IT$ and therefore

$$2jk(IT - I) - \left(\frac{mc_0}{\hbar}\right)^2 CIT = 0, \quad (9)$$

from which it follows

$$|T|^2 = \frac{1}{1 + [Cm^2c_0^2/(2k\hbar^2)]^2}. \quad (10)$$

This is very similar to the transmission loss of an acoustically thin homogeneous wall,

$$|T|^2 = \frac{1}{1 + [\omega m_{\mathrm{wall}}/(2\rho_0 c_0)]^2}, \quad (11)$$

where m_{wall} is mass of the wall per unit area and ρ_0 and c_0 are density and speed of sound in the surrounding air. After applying the relation, $\hbar\omega = mc_0^2$, one obtains

$$|T|^2 = \frac{1}{1 + [C\omega/(2c_0)]^2}. \quad (12)$$

Hence, the mass term $(mc_0/\hbar)^2(m_{\mathrm{wall}}/\rho_0)\delta(x_1)\phi$ in eq. (7) with $C = m_{\mathrm{wall}}/\rho_0$ is sufficient for describing sound transmission through a thin homogeneous wall, without additional constraints.

Spontaneous symmetry breaking in acoustic near field

The next simplest Lagrangian is defined for a complex scalar field ϕ. Since it has two degrees of freedom (real and imaginary part), we can make a constraint on one of them. It is usually imposed that the theory is symmetric under $\phi \to e^{j\alpha(x)}\phi$, where $\alpha(x)$ is an arbitrary phase not affecting the physically relevant part of the complex field, the amplitude. Notice that such a convention is different from taking the real part of the field as the physically relevant, which is more common in acoustics. The canonical Lorentz-invariant Lagrangian with the given symmetry $\phi \to e^{j\alpha}\phi$ reads

$$\mathcal{L} = \phi^{*,\mu}\phi_{,\mu} + \left(\frac{mc_0}{\hbar}\right)^2 \phi^*\phi, \quad (13)$$

with ϕ^* the complex conjugate of ϕ. However, the values of such a field at different points (and, therefore, derivatives and all associated mathematical expressions) depend on the (arbitrary) choice of $\alpha(x)$ for each particular x. In order to fix this, one has to introduce a connection [2], a vector gauge field $\bar{A}_\mu$ coupled to the scalar field ϕ. The Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) + \left(\frac{mc_0}{\hbar}\right)^2 \phi^*\phi, \quad (14)$$

which is the Lagrangian for scalar quantum electrodynamics. Here, $F_{\mu\nu} = \bar{A}_{\nu,\mu} - \bar{A}_{\mu,\nu}$, $D_\mu\phi = \phi_{,\mu} + je\bar{A}_\mu\phi/\hbar$ and $(D_\mu\phi)^* = \phi^*_{,\mu} - je\bar{A}_\mu\phi^*/\hbar$ are covariant derivatives, and e is a constant (elementary positive charge).

Analogously to the small metric perturbation $\bar{h}_{\mu\nu}$ considered in Ref. [5], $\bar{A}_\mu$ can represent an acoustic field due to a dipole source. It turns out that the Lagrangian in eq. (14) captures the interaction between sound waves and fluid particles or a dipole source in free space, with the particles represented by the complex scalar field ϕ. We observe spontaneous breaking of now continuous symmetry $\phi \to e^{j\alpha(x)}\phi$. The Lagrangian is [2]

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) \\ &- \left(\frac{mc_0}{\hbar}\right)^2 |\phi|^2 + \frac{\lambda}{4}|\phi|^4.\end{aligned} \quad (15)$$

Both λ and m can depend on certain parameter l. Moreover, $m^2 > 0$ when $l > L$, where L is a given point of symmetry breaking, and $m^2 < 0$ when $l < L$. In the latter case, the field is not a small perturbation of the background medium and this is the reason for replacing m^2 with positive $-m^2$ when compared to eq. (14).

The potential has a minimum for $\phi_0 = \sqrt{2m^2c_0^2/(\lambda\hbar^2)}e^{j\theta}$ and any $\theta(x)$. We can choose $\phi_0 = \sqrt{2m^2c_0^2/(\lambda\hbar^2)}$. Next we could expand $\phi = \phi_0 + \tilde{\phi}$. However, a more elegant approach is to take the two real degrees of freedom of $\tilde{\phi}(x)$, denoted with $\sigma(x)$ and $\pi(x)$, and write

$$\phi = \left(\phi_0 + \frac{\sigma}{\sqrt{2}}\right) e^{j\pi/(\sqrt{2}\phi_0)}. \quad (16)$$

After inserting this in eq. (15) and decoupling σ, which is irrelevant, we can let $m \to \infty$ and $\lambda \to \infty$ (keeping ϕ_0 unaffected), so that the Lagrangian without it becomes in the unitary gauge (choosing $\alpha(x)$ such that $\pi(x) = 0$) [2]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \left(\frac{e\phi_0}{\hbar}\right)^2 \bar{A}_\mu\bar{A}^\mu. \quad (17)$$

This is the Lorentz-invariant Proca Lagrangian for a massive gauge field with the mass $m = \sqrt{2}e\phi_0/c_0$. Indeed, the equation of motion which follows from this Lagrangian is the four-vector version of eq. (4) with $\bar{A}^\mu$ instead of ϕ and $\bar{A}^\mu{}_{,\mu} = 0$. Thus, the gauge field associated with acoustics, $\bar{A}_\mu$, becomes massive through the Higgs mechanism for $l < L$ (in the near field of the fluid particles acting as sources). If the positive charge e is analogous to the mass $\rho_0 dV$, where ρ_0 is the fluid density and $dV \sim L^3$, then $\phi_0 = c_0/\sqrt{2}$. In the quantum field theory, the field $\pi(x)$ (often referred to as pion) represents a Goldstone boson and the field $\sigma(x)$ is known as Higgs boson.

Acoustic Planck scale and Hawking radiation

The smallest length scale $L = \hbar/(mc_0)$ from eq. (6) at which the spontaneous symmetry breaking occurs is the scale of the stochastic motion of the fluid particles, the mean free path. It can be associated with a certain maximum frequency $\omega = c_0/L$. On the other hand, at such a small scale we can observe the stochastically moving particles as an aeroacoustic source (turbulence), as in Ref. [5]. Equation (19) there relates mass with the acoustic Schwarzschild radius L: $m = Lc_0^2/(2G)$, with G the gravitational constant. Equating the two length scales and masses, we find

$$L = \sqrt{\frac{2\hbar G}{c_0^3}} = \sqrt{2}L_P, \tag{18}$$

where $L_P = \sqrt{\hbar G/c_0^3}$ is the acoustic Planck length. A very small fluid element with volume $dV \sim L^3$ can be treated as a Planck particle. Within its volume, masses of the acoustic field and fluid particles field become equal. This is the smallest source region in which the massive acoustic spacetime perturbations cannot be distinguished from the moving particles of fluid and the separation on the left- and right-hand side in the Einstein field equations becomes inappropriate.

Thermodynamic analogy provides another interesting relation. The equation of state of an ideal gas gives

$$T_0 = \frac{Mc_0^2}{\gamma k_B}, \tag{19}$$

where γ is heat capacity ratio, k_B is the Boltzmann constant, T_0 is the reference temperature due to the stochastic motion of the molecules, and M is mass of a single molecule of the gas. This is the expression for Planck temperature if we adopt for the Planck mass

$$m_P = \sqrt{\frac{\hbar c_0}{G}} = \sqrt{\frac{Mc_0^3}{\omega G}} = M\sqrt{\frac{2c_0}{\omega L}} = \frac{M}{\gamma}. \tag{20}$$

Here we used the definitions of $\hbar$ and G. Therefore, the analogy holds for $\sqrt{\omega L/(2c_0)} \to \gamma$ and γ indicates acoustic compactness of the gas molecule. In air at room temperature, $\gamma = 1.4$ corresponds to the Helmholtz number value 3.92, which roughly separates the acoustic near and far field, and Planck mass closely corresponds to the molecular mass.

If we again observe motion of the molecules as a micro turbulence, a weak quadrupole source, we expect the increase of its efficiency in the vicinity of compact bodies. The amplification factor next to a compact rigid body is [9] $c_0/(\omega L)$, where L is given by eq. (18). Introducing this in eq. (19) and using again $L = 2GM/c_0^2$ and $Mc_0^2 = \hbar\omega$, we can write

$$T_H = \frac{4\pi}{\gamma}\frac{\hbar c_0^3}{8\pi G k_B M}, \tag{21}$$

which is, apart from the factor $4\pi/\gamma$ equivalent to the expression for Hawking radiation of a black hole with mass M [3]. The more efficient dipole radiation at the rigid boundary corresponds to the photon generation at the event horizon of the black hole.

Currents as sources

Details of the source mechanism, such as fluid-body interaction, are often unimportant and we are interested only in the resulting acoustic field. The interaction terms in the Lagrangian can then be replaced with the current following from the Lagrangian symmetry and Noether's theorem. In order to demonstrate this, we can first calculate derivative of the Lagrangian in eq. (13), which represents the field of fluid particles, with respect to α [2]:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\alpha} = 0 &= \sum_{n=1}^{2}\left(\frac{\partial\mathcal{L}}{\partial\phi_n}\frac{\partial\phi_n}{\partial\alpha} + \frac{\partial\mathcal{L}}{\partial(\phi_{n,\mu})}\frac{\partial(\phi_{n,\mu})}{\partial\alpha}\right)\\ &= \sum_{n=1}^{2}\left[\frac{\partial\mathcal{L}}{\partial\phi_n}\frac{\partial\phi_n}{\partial\alpha} + \left(\frac{\partial\mathcal{L}}{\partial(\phi_{n,\mu})}\frac{\partial\phi_n}{\partial\alpha}\right)_{,\mu} - \left(\frac{\partial\mathcal{L}}{\partial(\phi_{n,\mu})}\right)_{,\mu}\frac{\partial\phi_n}{\partial\alpha}\right],\end{aligned} \tag{22}$$

where $\phi_1 = \phi$ and $\phi_2 = \phi^*$. The first and the last term cancel according to eq. (2) and the remaining part gives the conservation law, $J^\mu{}_{,\mu}$, for the Noether current

$$J^\mu = \sum_{n=1}^{2}\frac{\partial\mathcal{L}}{\partial(\phi_{n,\mu})}\frac{\partial\phi_n}{\partial\alpha} = j\phi\phi^{*,\mu} - j\phi^*\phi^{,\mu}. \tag{23}$$

The Lagrangian in eq. (14) gives for the field $\bar{A}^\mu$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e\bar{A}^\mu J_\mu/\hbar \tag{24}$$

to the lowest order. This is the electromagnetic Lagrangian with the interaction expressed in terms of the current J_μ. Similarly, the Lagrangian which gives the quadrupole equation of motion ($\Box\bar{h}^{\mu\nu} = -2kGT^{\mu\nu}/c_0^4$ [5]) is [2]

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\bar{h}^{\mu\nu}\Box\bar{h}_{\mu\nu} - \bar{h}^{\mu\nu}\bar{h}_{\nu\alpha,\mu}{}^{,\alpha} + \bar{h}^\alpha{}_\alpha\bar{h}_{\mu\nu}{}^{,\mu\nu}\\ &- \frac{1}{2}\bar{h}^\alpha{}_\alpha\Box\bar{h}^\alpha{}_\alpha - \frac{2kG}{c_0^4}T_{\mu\nu}\bar{h}^{\mu\nu}.\end{aligned} \tag{25}$$

It is the leading order approximation of the Einstein-Hilbert Lagrangian with added current. The current is stress-energy tensor $T_{\mu\nu}$ and the interaction constant is $2kG/c_0^4$, where k is a dimensionless constant.

Stress-energy tensor is the current obtained when the parameter α is replaced with the coordinates x^ν, reflecting the symmetry of the action (not Lagrangian) under global spacetime translations. Equation (22) takes the form

$$\mathcal{L}^{,\nu} = \sum_{n=1}^{2}\left[\frac{\partial\mathcal{L}}{\partial\phi_n}\phi_n{}^{,\nu} + \left(\frac{\partial\mathcal{L}}{\partial(\phi_{n,\mu})}\phi_n{}^{,\nu}\right)_{,\mu} - \left(\frac{\partial\mathcal{L}}{\partial(\phi_{n,\mu})}\right)_{,\mu}\phi_n{}^{,\nu}\right], \tag{26}$$

which in general does not equal zero. However, eq. (2) applies and

$$\tau^{\mu\nu}{}_{,\mu} = \sum_{n=1}^{2}\left(\frac{\partial\mathcal{L}}{\partial(\phi_{n,\mu})}\phi_n{}^{,\nu}\right)_{,\mu} - \eta^{\mu\nu}\mathcal{L}_{,\mu} = 0, \tag{27}$$

where $\tau^{\mu\nu}$ is the conserved canonical stress-energy tensor and $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric. $\tau^{\mu\nu}$ can be brought to the symmetric $T^{\mu\nu}$ by transforming it as $\tau^{\mu\nu} \rightarrow \tau^{\mu\nu} + \Theta^{\alpha\mu\nu}{}_{,\alpha}$, with an appropriate $\Theta^{\alpha\mu\nu}$ satisfying the antisymmetry $\Theta^{\alpha\mu\nu} = -\Theta^{\mu\alpha\nu}$.

If boundaries are present, function f, the argument of the Heaviside function, does depend on the coordinates x^ν. However, we are considering symmetry of the action not the Lagrangian, so we can multiply entire eq. (26) with $H(f)$, where $f(t,\vec{x}) = 0$ at the surface of the body which bounds the spacetime, $f(t,\vec{x}) < 0$ inside the body, and $f(t,\vec{x}) > 0$ outside the body. This leads to

$$H(f)T^{\mu\nu}{}_{,\mu} = 0, \tag{28}$$

which comes down to the Ffowcs Williams and Hawkings aeroacoustic analogy [7], since $T^{\mu\nu}{}_{,\mu} = 0$ expresses the mass and momentum conservation in low Mach number flows [5]. The boundary moves with the velocity $\vec{u}$ and $\partial H/\partial t + \vec{u}\cdot\nabla H = 0$, which violates Lorentz invariance.

Following Ffowcs Williams and Hawkings,

$$HT^{\alpha\beta}{}_{,\beta} = (HT^{\alpha\beta})_{,\beta} - T^{\alpha\beta}H_{,\beta} = 0. \tag{29}$$

The first term is free space quadrupole and the second term is the dipole contribution which gives

$$\int T^{\alpha\beta}H_{,\beta}d^3\vec{x} = \oint_S (T^{\alpha j} - T^{\alpha 0}u^j/c_0)n_j d^2\vec{x}. \tag{30}$$

Here, n_j (with $j = 1...3$) is unit vector normal to S pointing outwards. A surface source in the form of the four-vector current is given by

$$\begin{aligned} J^\alpha &= T^{\alpha j}n_j/c_0 - T^{\alpha 0}u^j n_j/c_0^2 \\ &= [\rho(\vec{v}-\vec{u})\cdot\vec{n}, \rho\vec{v}(\vec{v}-\vec{u})\cdot\vec{n}/c_0 + p\vec{n}/c_0] \end{aligned} \tag{31}$$

and the divergence equals

$$\begin{aligned} J^\alpha{}_{,\alpha} &= \frac{\partial}{\partial t}\left(\rho(\vec{v}-\vec{u})\cdot\vec{n}/c_0\right) \\ &\quad + \nabla\cdot\left(\rho\vec{v}(\vec{v}-\vec{u})\cdot\vec{n}/c_0 + p\vec{n}/c_0\right). \end{aligned} \tag{32}$$

The first term describes thickness noise and the second term loading noise. For a rigid boundary ($\vec{u}\cdot\vec{n} = \vec{v}\cdot\vec{n}$) and $J^\alpha = [0, p\vec{n}/c_0]$. For a rigid, flat ($\nabla\cdot\vec{n} = 0$), and motionless ($\nabla p\cdot\vec{n} = 0$) boundary, $J^\alpha{}_{,\alpha} = 0$, corresponding to the conservation of charge in electromagnetism. In this way we have derived the Ffowcs Williams and Hawkings analogy from the variational principle.

Conclusion

In this work it is demonstrated that acoustics in fluids can be formulated using Lorentz-invariant Lagrangians and variational principle in acoustic spacetime. The analogy with electromagnetism and general relativity was utilized, which holds for low Mach number flows. The considered formalism allows a direct consideration of the acoustic phenomena of interest, which may be obscured by the generality of governing equations. From the didactic point of view, it points to the similarities between the Newtonian acoustics and modern field theories. Its advantages over the classical treatment for solving practical acoustic problems are yet to be investigated.

Further development of the theory can include replacing the complex fluid field with a spinor field, following the theory of quantum electrodynamics. Such a spinor field should be related to the aeroacoustic dipole source. Furthermore, the acoustic spacetime is built by the same fluid particles (represented by the complex fluid field ϕ) which constitute both the stress-energy tensor and the dipole current and couple to the acoustic gauge fields. In other words, in a simple fluid such as air, we deal with a pure (but perturbed) vacuum spacetime. The fluid particles are not immersed in the spacetime (as fermions in quantum field theory), they constitute it. This is conceptually very similar to the spin network of loop quantum gravity, which may provide further insight, for example a spinor description of a free space quadrupole source.

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