Clustering in an array of nonlinear and active oscillators as a model of spontaneous otoacoustic emissions

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Introduction

The mechanisms underlying the human ear's ability to detect very soft sounds and extremely small differences in sound intensity and frequencies across a wide dynamic range are still not fully understood. The high sensitivity can not be achieved by passive mechanisms, why the presence of an active mechanical feedback mechanism, the "active process", is often assumed in modelling studies [8]. This assumption is supported by the discovery of spontaneous otoacoustic emissions (SOAEs) [4]. SOAEs are sound signals that can be measured outside the ear in the absence of external stimuli, pointing towards self-sustained activity in the inner ear, possibly linked to outer hair cell motility [1]. An interesting characteristic of SOAEs are the narrow distributions of energy in the spectra of the emitted signal. SOAEs are vulnerable to a damage of outer hair cells and tend to disappear even for mild hearing impairment [6].

A model that has been of main interest in the investigation of the active process in the inner ear is a system of nonlinear, active, coupled limit cycle oscillators [7]. These models can account for many key aspects of the phenomena observed in SOAEs, including effects of entrainment and synchronisation. Even though systems of coupled oscillators and SOAEs are well investigated, the methods to quantify and describe the phenomena observed in theoretical studies of coupled oscillators and experimental studies of SOAEs differ. Because only one state variable is available for SOAEs (the pressure in the ear canal) the assumption of a periodic oscillation is commonly made, even though nonlinear and active oscillators often shown non-harmonic dynamics. In this paper, a model- and analysis framework inspired by previous models of coupled, nonlinear, active oscillators is described to link phenomena of coupled, nonlinear oscillators to SOAEs.

Methods

The model

The modeled system consisted of $N = 100$ Van der Pol oscillators, each described by the equation of motion:

$$\ddot{x} = -\omega^2 x + \mu(1 - x^2)\dot{x} \tag{1}$$

where x is the displacement from equilibrium, $\dot{x}$ and $\ddot{x}$ are the velocity and acceleration, respectively. The factor μ scales the nonlinearity of the system. To include tonotopy, a linear gradient in the eigenfrequency was introduced in the row of oscillators by

$$\omega_\mathrm{n} = \left(\omega_\mathrm{min} + \frac{(\mathrm{n}-1)\omega_\mathrm{max}}{N-1}\right) \cdot \epsilon_\mathrm{n} \tag{2}$$

$$\mathrm{n} \in \{1...N\} \qquad \epsilon_\mathrm{n} = \mathcal{N}(1, \nu) \tag{3}$$

The value of ω_n was varied stochastically for each oscillator by a random variable drawn from a normal distribution $\mathcal{N}$ with mean one and standard deviation ν to aid generation of clusters [5].

The force between two coupled neighbouring oscillators is given by

$$F_\mathrm{n} = d_{\mathrm{n}+1}(\dot{x}_{\mathrm{n}+1} - \dot{x}_\mathrm{n}) + k_{\mathrm{n}+1}(x_{\mathrm{n}+1} - x_\mathrm{n}) \tag{4}$$

$$+ d_\mathrm{n}(\dot{x}_{\mathrm{n}-1} - \dot{x}_\mathrm{n}) + k_\mathrm{n}(x_{\mathrm{n}-1} - x_\mathrm{n}) \tag{5}$$

where the dissipative and reactive coupling are described by d and k, respectively. Positions x_0 and x_{N+1} were held constant to zero.

Because the phase plane trajectory of a Van der Pol oscillator varies both with μ and ω, a scaling of the time differential was introduced by a scaling of μ to keep the ratio ω/μ constant and to result in identical phase trajectories of all N oscillators. Combining (1), (4) and an external driving force, F_ext, yields a set of first order differential equations for numerical treatment:

$$\frac{dx_\mathrm{n}}{dt} = \omega_\mathrm{n}\dot{x}_\mathrm{n} \tag{6}$$

$$\frac{d\dot{x}_\mathrm{n}}{dt} = -\omega_\mathrm{n}x_\mathrm{n} + \omega_\mathrm{n}\mu(1 - x^2)\dot{x}_\mathrm{n} + F_\mathrm{n} + F_\mathrm{ext} \tag{7}$$

Analysis

Steady state criteria

The dynamics of the system was simulated for time intervals of length T_int and stopped as soon as a steady state of the system was reached. The energy of each oscillator was approximated by:

$$E_\mathrm{n}(m) = \sum_{t=mT_\mathrm{int}}^{(m+1)T_\mathrm{int}} x_\mathrm{n}(t)^2 \;,\; m = 0, 1, 2... \tag{8}$$

If $E_\mathrm{n}(m+1)$ did not differ from $E_\mathrm{n}(m)$ by more than ten percent for all n, the system was defined to be in steady state.

Frequency, cluster and phase analysis

Three methods were used to analyse the frequency of each oscillator: 1) "Fourier mean method" ($\Omega_\mathrm{f,mean}$),

computed as the weighted mean of all frequencies Ω_i in the Fourier domain, 2) "Fourier peak method" ($\Omega_{\mathrm{f,peak}}$), computed by localising the frequency with the highest normalised peak m' in the Fourier domain, and 3) the "Zero crossing mean method" ($\Omega_{\mathrm{zc,mean}}$) where the frequency was determined from the inverse of the mean distance between zero crossings Δt_{zc} in the time domain.
Two rules for cluster localisation in the system are set up. Firstly, steps in frequency between oscillators considered to be in the same cluster should be smaller than half the size of the steps if all steps between the highest and lowest frequency were the same size. Secondly, a cluster consists of minimum 4 oscillators.
The phase coherence of the oscillators was computed using the estimated oscillation period combined with the amplitude and velocity of each oscillator. The normalised [displacement, velocity]-vectors after each oscillation period were added in the phase plane and divided by the number of the oscillation periods.

Numerical evaluation

The coupled equations of motion were solved using MATLAB's built in `ODE45` solver. For all simulations, values in (3) were set to

$$N = 100\ ,\ \omega_{\min} = 2\pi\ ,\ \omega_{\max} = 5 \cdot 2\pi \qquad (9)$$

The sampling frequency was set to $f_s = 1000\text{Hz}$ and the maximum step size used by `ODE45` to 0.01s. The simulation was run in intervals of length $T_{\mathrm{int}} = 20\text{s}$ until the system met the steady state criteria or until the maximum simulation time of 1000s was reached. All oscillators had initial displacement and velocity equal to 0 and were excited by a short rectangular pulse at time $t = 0\text{s}$ with width $w = 0.05\text{s}$ and unity height.
The systems were analysed for parameter combinations $\underline{P} = \{d, k, \mu, \nu\}$. To get a more explicit understanding, the systems were coupled by either dissipative or reactive coupling elements of varying strength. The following parameters were used for the simulations:

$$d \in \{0, 1, 2, 5, 10, 20, 50, 100\}, \qquad (10)$$
$$k \in \{0, 1, 2, 5, 10, 20, 50, 100\}, \qquad (11)$$
$$\mu \in \{0.5, 1, 1.5\ \ldots\ 4, 4.5, 5\}, \qquad (12)$$
$$\nu \in \{0, 0.1\} \qquad (13)$$

In the absence of stochastic variations of ω_{n} (i.e., $\nu = 0$) the system was simulated once, otherwise 25 times. The system with dissipative coupling $\underline{P_d} = \{100, 0, 1, 0\}$ and the system with reactive coupling $\underline{P_k} = \{0, 20, 1, 0\}$ will be described in more detail since these systems showed the most comprehensive clustering behaviour.
To analyse synchronisation effects by an external driving force, a force $F_{\mathrm{ext}} = \alpha \sin(2\pi f t)$ was applied to all oscillators, with α the amplitude and f the frequency of the sinusoidal force.

Results

Dissipative coupling

Figure 1A shows the spectral analysis of system $\underline{P_d}$ in steady state (reached after 80s of simulation, see Figure 1B). Clusters appeared at frequencies between 1 Hz to 5.5 Hz with equal spacing and size. This representation shows harmonics of each oscillator at frequencies of neighbouring clusters. The clusters overlap at the cluster edges, indicating two dominant frequencies for these oscillators.
When extracting clusters using $\Omega_{\mathrm{f,peak}}$ (Figure 1C), all

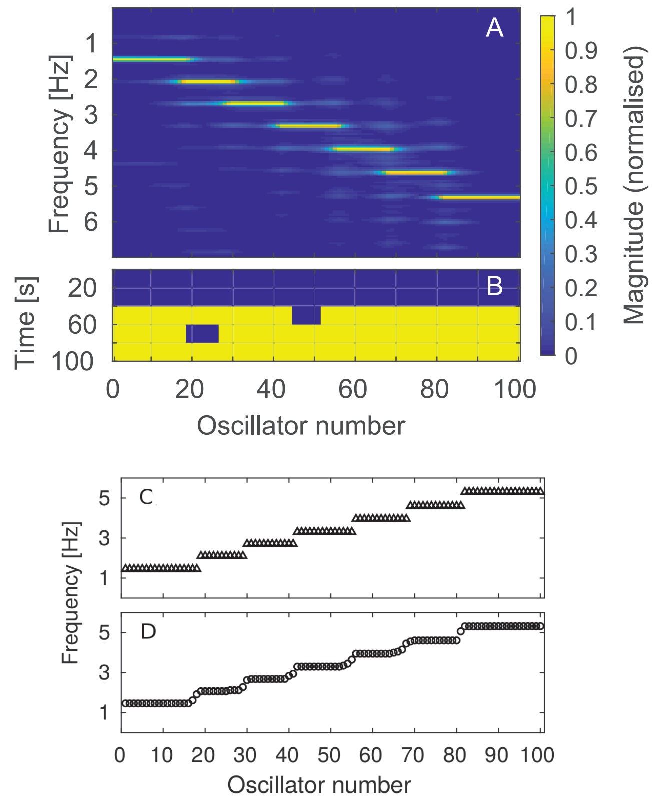

Figure 1: System $\underline{P_d}$. A) Spectral representation of the system dynamics. The spectrum for each oscillator was normalised to its maximum value. B) State of each oscillator as function of time. Steady state criteria are met for values of 1 and are not met for values of 0. C) Clusters found when analysed using $\Omega_{\mathrm{f,peak}}$. D) Clusters for analysis with $\Omega_{\mathrm{zc,mean}}$.

oscillators were assigned to a cluster. Analysis of the same data with $\Omega_{\mathrm{zc,mean}}$ (Figure 1D), resulted in a collection of oscillators that was not discrete but more smooth with some oscillators in between clusters, consistent with [5]. The $\Omega_{\mathrm{f,peak}}$ only selects the dominating frequency in the spectrum, while the $\Omega_{\mathrm{zc,mean}}$ detects the whole spectrum of frequencies. For oscillators in the bordering region between two clusters, two frequencies can be almost equally strong in the spectrum, which will only be caught by the $\Omega_{\mathrm{zc,mean}}$.
The number and size of the found clusters depended on coupling strength (d_n), nonlinearity scaling μ and the frequency analysis method. Larger clusters were found for stronger coupling and for higher degrees of nonlinearity. The mean cluster size and the synchronisation were slightly larger when using $\Omega_{\mathrm{f,peak}}$ compared to $\Omega_{\mathrm{zc,mean}}$.
The dynamics of the oscillators within a cluster or at the corners of a cluster are shown in Figure 2 for the $\Omega_{\mathrm{zc,mean}}$ frequencies found for system $\underline{P_d}$. The phase coherence in the middle of clusters is much higher than

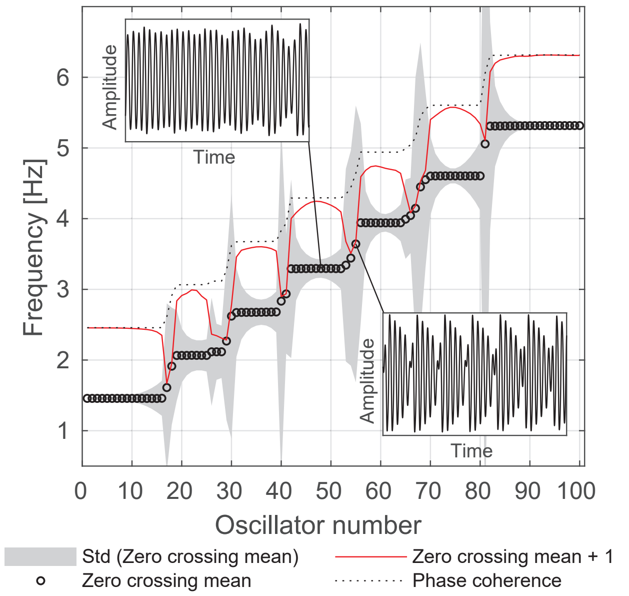

Figure 2: System $\underline{P_d}$, analysed in steady state. Analysis of the phase coherence by using the zero crossing mean method. The standard deviation of the frequencies is highest for oscillators between clusters. Furthermore, the phase coherence is highest in the middle of clusters and close to zero at the steps between clusters.

at the borders, where the uncertainty is correspondingly higher. The temporal dynamics showed a clear beating of one oscillator located between two clusters, consistent with two dominant frequencies as indicated in Figure 1A.

Reactive coupling

Figure 3 shows the spectral analysis of a system using reactive coupling. Two big clusters with frequencies around $f_l = 3.5$Hz and $f_h = 9$Hz are apparent in Figure 3A. The cluster frequency around 9 Hz is higher than the maximum eigenfrequency in the uncoupled system, indicating a contribution of the coupling elements to the dominant frequency of the oscillators. In between the clusters, the oscillators show a complex distribution of energy and no consistent clustering behaviour. Consistent with the results using dissipative coupling, clusters tended to be larger for higher amount of coupling and higher degree of nonlinearity, but the behaviour was much more complex.

Synchronisation with harmonic driving force

To analyse phase relations before and after synchronisation, the central cluster of system $\underline{P_d}$ was used. This cluster consisted of 11 oscillators with a mean frequency of 3.3 Hz. The phase analysis of oscillators in the cluster is shown in the absence (A) and presence (B) of an external driving force in Figure 4. In Figure 4A, the phase is shown for each time period $p = 1/3.3$s for each oscillator. The first and last oscillators showed a much wider spread of phases compared to the central oscillator. The spread of phase is consistent with the estimate of the phase coherence. Furthermore, the mean of the phase distribution

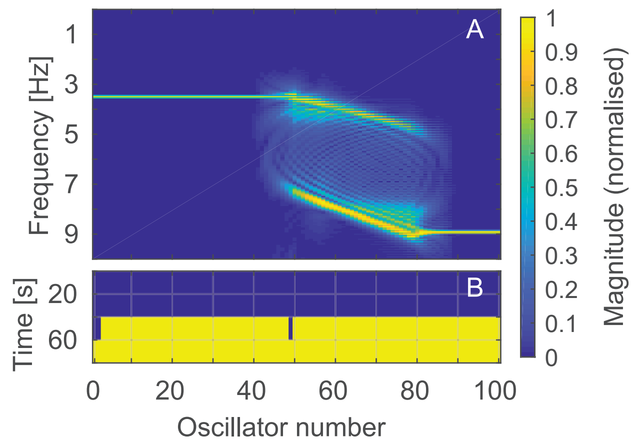

Figure 3: Same as Figure 1, but for system $\underline{P_k}$ with reactive coupling.

changed along the oscillators, indicating phase shift.
In the presence of an external driving force with frequency 3.25 Hz and amplitude 10, the cluster grows in size and the phase distributions of all oscillators become narrower. Especially in the middle of the cluster, the phases collapse into a very narrow region around the mean, indicating a very stable oscillation period. Neighbouring oscillators show a slight phase shift, similar to the result in the absence of the external driving force.

Discussion

For systems with dissipative coupling, similar cluster patterns compared to both [8, 3, 2] appeared in the systems. Although no frequency analysis method has been defined by [8], their results show a high similarity to the results in the current study using $\Omega_{\mathrm{f,peak}}$. The result that stronger coupling results in larger clusters can be explained by assuming that the interaction across oscillators happens exclusively through this parameter. Given that clustering is an effect of active, nonlinear oscillators, it is also intuitive that increasing the amount of nonlinearity increases the effect of clustering, leading to larger clusters. Regarding the reactive coupling, similarities to the data presented by [2] can be observed in the spectral representation in terms of some clearly defined clusters and some more blurry areas between the clusters. Consistent with this, [3] found that the number of clusters decreases for bigger values of the reactive coupling coefficient, which might be connected to the increasing mean cluster size documented in this paper.
The often neglected information of oscillator clusters is the phase relation across oscillators within one cluster. With the results in Figure 4 it becomes clear that even though oscillators might be entrained to a specific frequency, the local properties of the oscillators will have an effect on the phase relation within a cluster. For phenomena that represent the compound result of a whole cluster like SOAEs measured in the ear canal, this phase shift will have important consequences on the measured magnitude. The measured magnitude can hence not be used to estimate the size of the cluster because the ad-

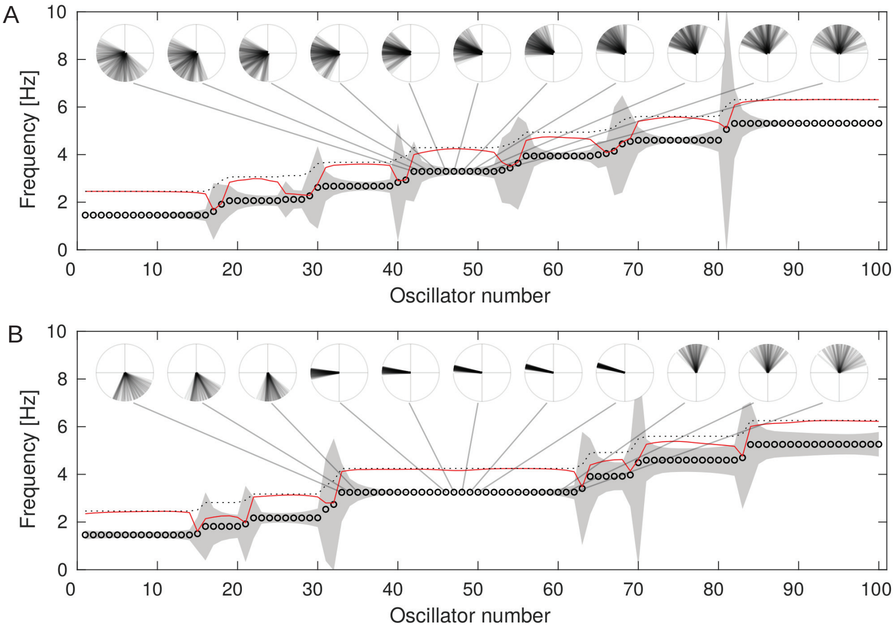

Figure 4: A) Phase analysis of the central cluster in the system $\underline{P_d}$ in steady state. The cluster consisted of oscillators $n = 42$ to $n = 52$. The mean frequency of the cluster was 3.3 Hz. The phase of each oscillator was investigated by marking points in phase space for each time period $p = 1/3.3$ s. Same legend as in 2E. B) Phase analysis of central cluster in the system $\underline{P_d}$ when in steady state while being synchronised by external force with frequency $f = 3.25$ Hz and amplitude $\alpha = 10$. The cluster consisted of oscillators $n = 33$ to $n = 62$.

dition of phase-delayed components will reduce the resulting compound amplitude, in the extreme case to zero for components in anti phase. Phase distributions are, in general, more narrow for oscillators located centrally in a cluster than for oscillators remote from the cluster center. This might be connected to frequency difference between the cluster frequency and the eigenfrequency of the local oscillator. An external driving force heavily reduces the width of the phase distributions, making them very narrow for oscillators in the center of the cluster. It is, however, interesting to note that the phase shift of oscillators within a cluster remains the same as in the absence of the external driving force.

Conclusions

A system of coupled Van der Pol oscillators with a frequency gradient showed clustering effects for both dissipative and reactive coupling. The number of and distance between clusters is dependent on the magnitude and type of the coupling between oscillators. Hence, the distance observed between SOAEs might provide information about the type and strength of coupling in the inner ear. The size and frequency of clusters depends, however, on the metric used to quantify the oscillation period of each oscillator. Finally, oscillators within one cluster showed a linear phase gradient. Hence, a better understanding of the parameters defining this phase gradient and the interaction with the size and frequency distance of clusters is required to be able to link these properties to analogue characteristics observed in SOAEs.

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References

[1] J. Ashmore. Cochlear outer hair cell motility. *Physiological Reviews*, 88(1):173–210, 2008.

[2] B. Epp, H. Wit, and P. van Dijk. Clustering of cochlear oscillations in frequency plateaus as a tool to investigate soae generation. *Aip Conference Proceedings*, 1703(1):090025, 2016.

[3] M. Gelfand, O. Piro, M. O. Magnasco, and A. J. Hudspeth. Interactions between hair cells shape spontaneous otoacoustic emissions in a model of the tokay gecko's cochlea. *Plos One*, 5(6):e11116, 2010.

[4] D. Kemp. Stimulated acoustic emissions from within human auditory-system. *Journal of the Acoustical Society of America*, 64(5):1386–1391, 1978.

[5] G. V. Osipov, J. Kurths, and C. Zhou. Synchronization in oscillatory networks. 2007.

[6] R. Probst, B. Lonsbuty-Martin, and G. Martin. A review of otoacoustic emissions. *Journal of the Acoustical Society of America*, 89(5):2027–2067, 1991.

[7] C. Talmadge, A. Tubis, H. Wit, and G. Long. Are spontaneous otoacoustic emissions generated by self-sustained cochlear oscillators. *Journal of the Acoustical Society of America*, 89(5):2391–2399, 1991.

[8] A. Vilfan and T. Duke. Frequency clustering in spontaneous otoacoustic emissions from a lizard's ear. *Biophysical Journal*, 95(10):4622–4630, 2008.