PROCEEDINGS of the
23rd International Congress on Acoustics

9 to 13 September 2019 in Aachen, Germany

On the stability of Gabor phase retrieval

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ABSTRACT

Phase retrieval refers to the problem of recovering a signal from phaseless measurements. Gabor phase retrieval, in particular, is concerned with reconstruction from the absolute value of the Gabor transform and has applications in the time-frequency analysis of audio signals. From a mathematical point of view, phase retrieval (from frame coefficients) is a challenging problem as it has been shown to be unstable in infinite dimensional Hilbert spaces and severely ill-conditioned in finite dimensional spaces. However, it has also been shown that one can relax the classical stability regime to so-called semi-global phase reconstruction and obtain a stability result for phase retrieval from the continuous Gabor transform in this setting. Recently, we were able to adapt semi-global phase reconstruction to the discrete case and to prove a promising stability result for the discrete Gabor transform. In this contribution, we survey selected highlights from recent research on phase retrieval from frame coefficients with emphasis on phase retrieval from Gabor measurements. In particular, we review results on semi-global stability of phase retrieval in the infinite dimensional case and present our result in the finite dimensional setting.

Keywords: Phase retrieval, Time-frequency analysis, Gabor transform

1 INTRODUCTION

In audio processing, among other applications, one sometimes wants to reconstruct an unknown signal x from measurements that do not contain phase information. This reconstruction problem is widely known as *phase retrieval*. Mathematically, phase retrieval can be modelled with the help of a Hilbert space $\mathscr{H}$ and a frame $\Phi = \{\phi_\lambda\}_{\lambda\in\Lambda}$ of $\mathscr{H}$ in such a way that we seek to reconstruct $x \in \mathscr{H}$ from the measurements $\{|(x,\phi_\lambda)|\}_{\lambda\in\Lambda}$. In this form, it occurs in applications such as signal noise reduction (1), automatic speech recognition (2) or the phase vocoder (3, 4).

The most popular algorithms for phase retrieval up to this point are so-called alternating projection methods originating with the well-known Gerchberg–Saxton algorithm (5) and the further developments in (6). Unfortunately, it is known that the Gerchberg–Saxton algorithm can fail to converge to the desired solution and the same is true for the methods proposed in (6) in the presence of noise. Of course, various methods exist to overcome the limitations of the algorithms in (5, 6), but as we do not want to focus on algorithmic methods in this paper, we will refer the reader to the survey (7). In recent years, a host of other methods have surfaced which we want to mention, however. Among those are PhaseLift (8), which solves the phase retrieval problem using semidefinite programming techniques but suffers from immense computational costs, Wirtinger Flow (9), which boils down to performing gradient descent on a non-convex cost function from a suitably chosen initial guess, and PhaseMax (10, 11), which like PhaseLift uses a convex relaxation but unlike PhaseLift operates on the natural parameter space and does therefore allow for efficient implementation.

Most of the theoretical guarantees for the success of the aforementioned methods rely on measurement systems $\{\phi_\lambda\}_{\lambda\in\Lambda} \subset \mathbb{C}^L$ whose individual elements are chosen independently at random and which contain sufficiently many measurement vectors (8, 9, 11). In addition, there are a few results based on random measurement systems which are more closely related to the Gabor transform (12, 13). Notably absent from these results, however, are statements about deterministic measurement systems and in particular frames for Hilbert spaces.

Among the theoretical work for phase retrieval from deterministic measurements, we want to shortly highlight the research on polarisation for phase retrieval in (14, 15) as well as the research on generic frames in (16–19). More theoretical work (and numerical experiments) on deterministic phase retrieval has been carried out in (20–24).

This paper is sructured as follows. Section 2 provides an overview over the classical mathematical theory of phase retrieval. In particular, the instability of phase retrieval in infinite dimensional spaces and the stability yet ill-conditionedness of phase retrieval in finite dimensional spaces are discussed. Section 3 contains the introduction of the concept of semi-global phase reconstruction. In particular, the pioneering work on infinite dimensional spaces (25) as well as recent work on finite dimensional spaces is described. The paper ends in a conclusion which points out interesting future avenues of research.

2 PHASE RETRIEVAL IN THE CLASSICAL SETTING

Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{R}$, $\mathscr{H}$ a Hilbert space over $\mathbb{K}$ with inner product $(\cdot,\cdot) : \mathscr{H} \times \mathscr{H} \to \mathbb{K}$, Λ some index set and $\Phi := \{\phi_\lambda\}_{\lambda\in\Lambda} \subset \mathscr{H}$ a set of measurement elements. Then, the recovery of $x \in \mathscr{H}$ from the measurements $\{|(x,\phi_\lambda)|\}_{\lambda\in\Lambda}$ is immediately seen to be impossible as all signals in $\{x\mathrm{e}^{\mathrm{i}\alpha}\}_{\alpha\in\mathbb{R}}$ generate the same measurements. It is the canon in the literature to circumvent this problem by recovering signals up to global phase. Mathematically, this is captured by introducing the quotient spaces $\mathbb{P}_\mathbb{R}\mathscr{H} := \mathscr{H}/\{-1,1\}$ as well as $\mathbb{P}_\mathbb{C}\mathscr{H} := \mathscr{H}/\mathscr{S}^1$ and measuring the distance between signals via the metric

$$d_\mathscr{H}(x,y) := \min\{\|x-\tau y\|_\mathscr{H} \mid \tau \in \mathbb{K},\ |\tau| = 1\}, \qquad x,y \in \mathscr{H}.$$

The phase retrieval problem is formulated as the inversion of the phase retrieval operator $\mathscr{A}_\Phi : \mathbb{P}_\mathbb{K}\mathscr{H} \to \mathbb{R}_+^\Lambda$ given by

$$\mathscr{A}_\Phi[x] := \{|(x,\phi_\lambda)|\}_{\lambda\in\Lambda}, \qquad x \in \mathscr{H}.$$

In this paper, we are mainly interested in phase retrieval from the discrete and continuous Gabor transform. We will therefore consider a window function $\phi \in L^2(\mathbb{R})$ and a signal $x \in L^2(\mathbb{R})$ in order to define the (continuous) Gabor transform

$$\mathscr{V}_\phi[x](t,\xi) := \int_\mathbb{R} x(s)\overline{\phi(s-t)}\mathrm{e}^{-2\pi\mathrm{i}s\xi}\,\mathrm{d}s, \qquad t,\xi \in \mathbb{R}.$$

Gabor phase retrieval does then refer to the recovery of x from $|\mathscr{V}_\phi[x]|$ which is exactly the recovery of x from $\mathscr{A}_\Phi[x]$, for $\Phi := \{\phi_{(t,\xi)}(s) = \phi(s-t)\mathrm{e}^{2\pi\mathrm{i}s\xi} \mid t,\xi \in \mathbb{R}\}$. The discrete case is very similar to the continuous one: For a window function $\phi \in \mathbb{C}^L$ and a signal $\mathbf{x} \in \mathbb{C}^L$, we introduce the discrete Gabor transform

$$\mathscr{V}_\phi^\mathrm{d}[\mathbf{x}](m,n) := \frac{1}{\sqrt{L}} \cdot \sum_{\ell=0}^{L-1} \mathbf{x}(\ell)\overline{\phi(\ell-m)}\mathrm{e}^{-2\pi\mathrm{i}\frac{\ell n}{L}}, \qquad m,n = 0,\ldots,L-1,$$

where the indexing of ϕ (and in general all vectors in this paper) is understood to be evaluated modulo L. Discrete Gabor phase retrieval does then refer to the recovery of $\mathbf{x}$ from $|\mathscr{V}_\phi^\mathrm{d}[\mathbf{x}]|$ which is exactly the recovery of $\mathbf{x}$ from $\mathscr{A}_{\Phi_\mathrm{d}}[\mathbf{x}]$, for $\Phi_\mathrm{d} := \{\phi_{(m,n)}(\ell) = \phi(\ell-m)\mathrm{e}^{2\pi\mathrm{i}\ell n/L} \mid m,n = 0,\ldots,L-1\}$.

2.1 Injectivity

The injectivity of the phase retrieval operator is an interesting (and vast) topic in its own right. We broach this subject here as it plays an important role in the stability results we present subsequently. It is well-known (16, 17, 26, 27) that injectivity of the phase retrieval operator in spaces over the real field is equivalent to the so-called complement property introduced by Balan, Casazza and Edidin (16) in 2006. In spaces over the complex field, in contrast, the complement property is merely necessary for injectivity.

For Gabor phase retrieval, one can derive quite palpable uniqueness guarantees based on the relation of the Gabor measurements with the ambiguity functions $\mathscr{V}_x[x]$ and $\mathscr{V}_\phi[\phi]$ (22, 23, 28, 29). In particular, the following simple lemma holds (28).

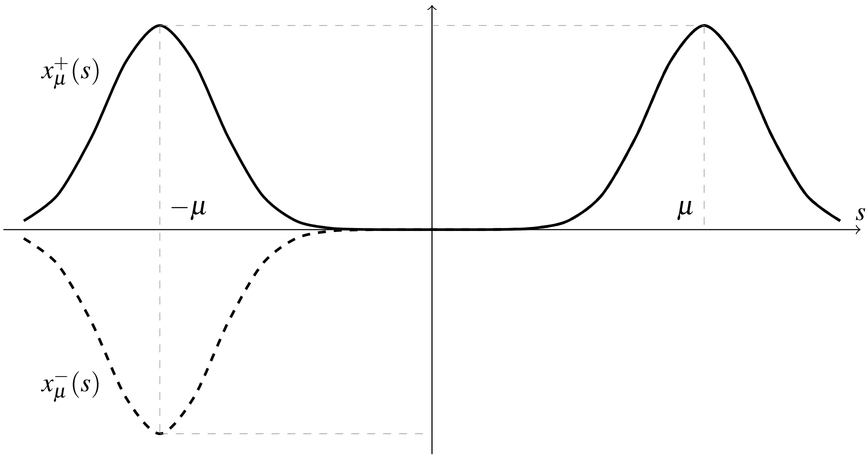

Figure 1. The most prominent example for instability of phase retrieval with continuous Gabor measurements.

Lemma 2.1. *Let $x, \phi \in L^2(\mathbb{R})$ with*

$$\mathscr{V}_\phi[\phi](t,\xi) \neq 0, \qquad t,\xi \in \mathbb{R}.$$

Then, x can be uniquely recovered from $|\mathscr{V}_\phi[x]|$ (up to global phase).

For the discrete Gabor transform, one can derive the same result (see (23)).

2.2 Stability

The stability of phase retrieval in the classical setting can be fully characterised by continuity properties of the inverse of the phase retrieval operator. A first stability result, often called weak stability, is proven in (26, 27).

Lemma 2.2. *Let $\Phi = \{\phi_\lambda\}_{\lambda\in\Lambda}$ be a frame for $\mathscr{H}$ and suppose that $\mathscr{A}_\Phi$ is injective. Then, $\mathscr{A}_\Phi^{-1}$ is continuous.*

Mere continuity of $\mathscr{A}_\Phi^{-1}$ is usually not enough, however, as it does not allow us to quantify the stability constant of the problem, which is the smallest $c > 0$ such that for all $x, y \in \mathscr{H}$,

$$d_{\mathscr{H}}(x,y) \leq c \cdot \|\mathscr{A}_\Phi[x] - \mathscr{A}_\Phi[y]\|, \tag{1}$$

where $\|\cdot\|$ is some suitable norm. Using the compactness of the unit ball in finite dimensions, the authors of (26, 27) are able to show that phase retrieval in finite dimensional Hilbert spaces always admits a finite stability constant. We say that finite dimensional phase retrieval is (strongly) stable.

Theorem 2.3. *Let $\mathscr{H}$ be finite dimensional, $\Phi = \{\phi_\lambda\}_{\lambda\in\Lambda}$ a frame for $\mathscr{H}$ and suppose that $\mathscr{A}_\Phi$ is injective. Then, the stability constant of phase retrieval satisfies $c < \infty$.*

Unfortunately, the picture looks very different in the infinite dimensional setting and in fact the authors of (26, 27) proved that infinite dimensional phase retrieval is never (strongly) stable.

Theorem 2.4. *Let $\mathscr{H}$ be infinite dimensional. Then, for every frame $\Phi = \{\phi_\lambda\}_{\lambda\in\Lambda}$ for $\mathscr{H}$ (with finite upper frame bound) and every $c > 0$, there exist $x, y \in \mathscr{H}$ such that inequality* (1) *is violated.*

2.3 Severe ill-posedness and ill-conditionedness

Finite dimensional phase retrieval is a challenging problem in spite of its stability. This is because the stability constants of phase retrieval from frame coefficients in finite dimensional Hilbert spaces might increase drastically

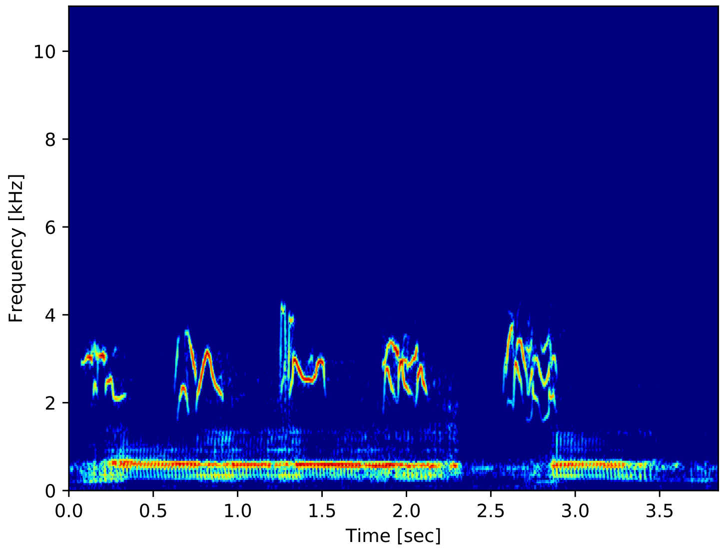

Figure 2. Spectrogram of an audio signal in which one may hear a bird and a bison.

in the dimension of the Hilbert spaces in question. This was first noticed by Cahill, Casazza and Daubechies (26) in 2016 when they constructed a sequence of finite dimensional subspaces (and frames) of an infinite dimensional space along which the stability constants increased exponentially. A very similar construction that is easily understood can be found in (31). The authors consider two signals $x_\mu^\pm \in L^2(\mathbb{R})$, $\mu > 0$, consisting of two Gaussian bumps each. More precisely, they use the Gaussian window $\phi(s) = \exp(-\pi s^2)$ and the signals

$$x_\mu^+(s) := \phi(s-\mu) + \phi(s+\mu), \qquad x_\mu^-(s) := \phi(s-\mu) - \phi(s+\mu)$$

depicted in figure 1. It is not hard to see that $d_{L^2(\mathbb{R})}(x_\mu^+, x_\mu^-) = 2^{3/4}$ and through some technical calculations one can establish (see (31)) that there exists a constant $c > 0$ such that

$$\left\| |\mathscr{V}_\phi[x_\mu^+]| - |\mathscr{V}_\phi[x_\mu^-]| \right\|_{H^1(\mathbb{R})} \leq c\mathrm{e}^{-\mu^2},$$

where $\|\cdot\|_{H^1(\mathbb{R})}$ denotes the Sobolev norm. In particular, the stability of the recovery of $x_\mu^\pm$ from Gabor measurements degrades at a rate that is (at least) exponential in the size of the time gap squared μ^2 and we say that phase retrieval from continuous Gabor measurements is severely ill-posed. In the discrete case, one can derive analogous results. In particular, it is true that if we consider a family of window functions $\{\phi_L \,|\, \phi_L \in \mathbb{C}^L,\ L \in \mathbb{N}\} \subset \bigcup_{L\in\mathbb{N}} \mathbb{C}^L$ which is exponentially decaying in the sense that there exist $L_0 \in \mathbb{N}$, $c > 0$, $\mu > 0$ such that for all $\ell, L \in \mathbb{N}$ with $L \geq L_0$ and $L_0 \leq \ell + 1 \leq L$, $|\phi_L(\ell)| \leq c\mathrm{e}^{-\mu\ell}$, then the stability constants $(c_L)_{L\in\mathbb{N}}$ of the corresponding discrete Gabor phase retrieval problems are (at least) exponentially increasing. We say that phase retrieval from discrete Gabor measurements is severely ill-conditioned.

3 SEMI-GLOBAL PHASE RECONSTRUCTION

It was noticed by Alaifari, Daubechies, Grohs and Yin (25) in 2018 that all examples that were designed in the literature to show the severe ill-posedness of phase retrieval in infinite dimensions or the severe ill-conditionedness of phase retrieval in finite dimensions had one thing in common: The underlying measurements $\mathscr{A}_\Phi[x] = \{|(x, \phi_\lambda)|\}_{\lambda\in\Lambda}$ were disconnected in the sense that one could decompose $(x, \phi_\cdot) : \Lambda \to \mathbb{K}$, $\lambda \mapsto (x, \phi_\lambda)$, into a sum of functions $F_k : \Lambda \to \mathbb{K}$, $k = 1,\dots,K$, whose support was essentially disjoint. Examples of this include the functions $x_\lambda^\pm \in L^2(\mathbb{R})$ designed in (31) and pictured in figure 1 which lead to a disconnectedness of the Gabor transform in time or the digital signal $\mathbf{x} \in \mathbb{C}^L$ in which birdsong is superimposed on the bellow of a bison and whose spectrogram (that is the absolute value of the discrete Gabor transform) is shown in figure 2. This insight led the authors of (25) to suggest that one should try to reconstruct signals up to semi-global phase and not up to global phase as was common in the literature up to then. In particular, they suggested that instead of reconstructing $(x,\phi.) = \sum_{k=1}^K F_k$ (up to global phase) from $|\sum_{k=1}^K F_k| = \mathscr{A}_\Phi[x]$, one should try to reconstruct $\sum_{k=1}^K \mathrm{e}^{\mathrm{i}\alpha_k} F_k$, with $\alpha_k \in \mathbb{R}$, $k = 1,\dots,K$. Remarkably, this suggestion ties in very nicely with a physical property of Gabor transform-based time-frequency analysis: The human ear cannot distinguish between audio signals differing by semi-global phase. For examples of this, the reader is strongly advised to visit (32). Mathematically, the authors of (25) formulated semi-global phase reconstruction by introducing so-called atoll functions and were able to prove the following stability result for them[1].

Theorem 3.1 (Theorem 1.5 in (25))**.** *Let $\phi(s) = \exp(-\pi s^2)$, $f = \sum_{k=1}^K f_k \in L^2(\mathbb{R})$ and $g = \sum_{k=1}^K g_k \in L^2(\mathbb{R})$, with*

$$\mathscr{V}_\phi[f_k]\big|_{B_{r_k}(z_k)} \in \left\{ F \in \mathscr{C}^1(B_{r_k}(z_k)) \,\middle|\, \delta_k \le |F(z)| \le \Delta_k,\ |\nabla |F|(z)| \le \Delta_k,\ z \in B_{r_k}(z_k) \right\},$$

where $r_k, \delta_k, \Delta_k > 0$ and $z_k \in \mathbb{C}$, as well as

$$\int_{\mathbb{R}^2 \setminus B_{r_k}} \left|\mathscr{V}_\phi[f_k](t,\xi)\right|^2 \mathrm{d}t\,\mathrm{d}\xi \le \varepsilon_k^2 \qquad \text{and} \qquad \int_{\mathbb{R}^2 \setminus B_{r_k}} \left|\mathscr{V}_\phi[g_k](t,\xi)\right|^2 \mathrm{d}t\,\mathrm{d}\xi \le \varepsilon_k^2,$$

where $\varepsilon_k > 0$, for $k = 1,\dots,K$. Then, there exist $c_0, c_1 > 0$ such that

$$\inf_{\alpha_1,\dots,\alpha_K} \sum_{k=1}^K \left\| f_k - \mathrm{e}^{\mathrm{i}\alpha_k} g_k \right\|_{L^2(\mathbb{R})} \le c_0 \cdot \sum_{k=1}^K \frac{\Delta_k^2}{\delta_k^2} \cdot (1 + c_1 r_k) \cdot \left(1 + c_1 (r_k + 1) \mathrm{e}^{\pi r_k^2/2}\right) \cdot \| |\mathscr{V}_\Phi[f]| - |\mathscr{V}_\Phi[g]| \|_{H^1(B_{r_k}(z_k))} + \sum_{k=1}^K \varepsilon_k.$$

The proof of the above result fundamentally depends on the fact that the Gaussian window generates a Gabor transform with analytic model space (this is due to the intimate relation of the Gabor transform with Gaussian window and the Bargmann transform which is explained in section 3.4 of (33)). Regrettably, it was shown in (34) that the Gaussian is the only window function for which the Gabor transform has an analytic model space. Because of this, it remains unknown whether theorem 3.1 may be extended to more general window functions $\phi \in L^2(\mathbb{R})$.

It is known, however, that theorem 3.1 may be extended to phase retrieval from discrete Gabor measurements when one adapts the notion of semi-global phase reconstruction in a suitable (graph theoretic) way (29): Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^L$ be two signals, $\delta_0 > 0$ and $\ell_0 \in \{0,\dots,\lfloor L/2 \rfloor - 1\}$. Define the graph $G = (V,E)$ by

$$V := \{\ell \in \{0,\dots,L-1\} \,|\, |\mathbf{x}(\ell)|, |\mathbf{y}(\ell)| > \delta_0\} \tag{2}$$

and placing an edge between two vertices $\ell, \ell' \in V$ if

$$1 \le \left|\ell - \ell'\right| \le \ell_0 + 1 \mod L. \tag{3}$$

In general, the graph G will consist of $K \in \{1,\dots,L-1\}$ connected components whose vertex sets can be denoted by $\{V_k\}_{k=1}^K$. Then, we say that $\mathbf{x}$ agrees with $\mathbf{y}$ up to *semi-global phase* if for every connected component $V_k \in \{V_k\}_{k=1}^K$, there exists an $\alpha_k \in \mathbb{R}$ such that for all $\ell \in V_k$,

$$\mathbf{x}(\ell) = \mathrm{e}^{\mathrm{i}\alpha_k} \mathbf{y}(\ell).$$

[1]Theorem 3.1 is displayed in a drastically simplified version here and the reader is invited to consult (25) for the full result and, in particular, an in-depth explanation of the terminology 'atoll domains'.

Theorem 3.2 (Theorem 3.5 in (29))**.** *Consider two signals* $\mathbf{x},\mathbf{y} \in \mathbb{C}^L$*, a window* $\phi \in \mathbb{C}^L$*, two tolerance parameters* $\delta_0,\delta_1 > 0$ *and a maximum time separation parameter* $\ell_0 \in \{0,\dots,\lfloor L/2\rfloor - 1\}$*. Let* $G = (V,E)$ *be defined by the equations* (2) *as well as* (3) *and assume that G has K connected components with vertex sets* $\{V_k\}_{k=1}^K$*. Then,*

$$\inf_{\alpha_1,\dots,\alpha_K\in\mathbb{R}} \sum_{k=1}^{K} \left\| \mathbf{x} - e^{i\alpha_k}\mathbf{y} \right\|_{\ell^2(V_k)} \le \frac{1}{\delta_0\delta_1} \cdot \left(\frac{1}{2} + \frac{\min\{\|\mathbf{x}\|_\infty, \|\mathbf{y}\|_\infty\}}{\delta_0} \cdot \sum_{k=1}^{K} |V_k| \right) \cdot \left\| \left|\mathcal{V}_\phi^{\mathrm{d}}[\mathbf{x}]\right|^2 - \left|\mathcal{V}_\phi^{\mathrm{d}}[\mathbf{y}]\right|^2 \right\|_{\mathrm{F}} + \frac{1}{\delta_0} \cdot \left(\frac{1}{2} + \frac{\min\{\|\mathbf{x}\|_\infty, \|\mathbf{y}\|_\infty\}}{\delta_0} \cdot \sum_{k=1}^{K} |V_k| \right) \cdot \varepsilon$$

holds, where $\|\cdot\|_{\mathrm{F}}$ *denotes the Frobenius norm and with* $\varepsilon > 0$ *satisfying*

$$\frac{\varepsilon^2}{2} = \sum_{m=0}^{\ell_0+1} \sum_{\substack{n=0 \\ |\mathfrak{A}_{\mathrm{d}}[\phi](m,n)|\le\delta_1}}^{L-1} |\mathfrak{A}_{\mathrm{d}}[\mathbf{x}](m,n) - \mathfrak{A}_{\mathrm{d}}[\mathbf{y}](m,n)|^2.$$

The main weakness of the above statement is that it only allows for the treatment of signals whose discrete Gabor transform is disconnected in time. A similar theorem may be deduced for signals with frequency gap but a unifying result in which general time-frequency disconnectedness is treated is not known.

We want to finish this section by remarking that there exists a numerical method (35) for phase retrieval from Gabor measurements which does signal reconstruction up to semi-global phase as described in (25). The method, called phase gradient heap integration, relies on the analyticity of the model space of the Gabor transform with Gaussian window and reconstructs phase locally based on a finite difference version of the Cauchy–Riemann equations.

4 CONCLUSIONS

In this survey paper, we have summarised selected highlights from recent research on Gabor phase retrieval. In particular, we have covered classical stability results for phase retrieval from frame coefficients, the notion of semi-global phase reconstruction introduced by Alaifari, Daubechies, Grohs and Yin as well as their stability result for continuous Gabor phase retrieval with Gaussian window. We want to end this survey by presenting two interesting open research questions:

1. Can one prove stability of continuous Gabor phase retrieval in the semi-global regime for a window function $\phi \in L^2(\mathbb{R})$ which is not the Gaussian? The difficulty here is that the approach by Alaifari, Daubechies, Grohs and Yin breaks down for any non Gaussian window as the Gabor transform fails to have an analytic model space.

2. Can one prove stability (in the sense that the stability constant does not increase too drastically in the space dimension) of discrete Gabor phase retrieval in a truly semi-global regime? In particular, what happens for general time-frequency disconnectedness in the discrete Gabor transform?

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