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Extensions of the Born approximation for acoustic radiation force and torque to inhomogeneous objects and progressive spherical waves

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Abstract

The Born approximation was used in recent work to develop simple expressions for the acoustic radiation force and torque acting on objects of arbitrary shape [Jerome et al., J. Acoust. Soc. Am. **145**, 36-44 (2019)]. Results for spheres and prolate spheroids in plane standing wave fields were compared with the full solution based on spherical harmonic expansions of the incident and scattered fields. The approximation was shown to be accurate for objects with material properties similar to those of the surrounding fluid, and with dimensions up to about one wavelength. Here, closed-form expressions based on the Born approximation are presented for a finite cylinder with material inhomogeneity that varies linearly along the axis of the cylinder. The expressions reveal the dependence of the radiation force and torque on the position and orientation of the inhomogeneous cylinder with respect to an incident plane standing wave. Next, the Born approximation is used to obtain a closed-form expression for the radiation force on a homogeneous compressible sphere as a function of its distance from the center of a diverging or converging spherical wave. The validity of this solution is assessed via comparison with the full solution.

Keywords: Acoustic radiation force, Born approximation

1 INTRODUCTION

Simple volume integrals based on the Born approximation were developed previously for the acoustic radiation force and torque exerted by a standing plane wave on a homogeneous object of arbitrary shape. [1] The principal restriction is that the material properties of the object are similar to those of the surrounding fluid, in which case the approximation is reasonably accurate for objects with dimensions up to about one wavelength. Results were presented for spheres, finite cylinders, and prolate spheroids.

Here the Born approximation is generalized as follows to describe the radiation force $\mathbf{F}$ and torque $\boldsymbol{\tau}$ acting on an inhomogeneous object of volume V in a multidimensional wave field:

$$\mathbf{F} = -\int_V [f_1(\mathbf{r})\boldsymbol{\nabla}\langle E_p\rangle - \tfrac{3}{2}f_2(\mathbf{r})\boldsymbol{\nabla}\langle E_k\rangle]\,dV \qquad \boldsymbol{\tau} = -\int_V \mathbf{r}\times[f_1(\mathbf{r})\boldsymbol{\nabla}\langle E_p\rangle - \tfrac{3}{2}f_2(\mathbf{r})\boldsymbol{\nabla}\langle E_k\rangle]\,dV \tag{1}$$

where $\langle E_p\rangle$ and $\langle E_k\rangle$ are the time-averaged potential and kinetic energy densities, respectively, in the incident sound field. Material properties are taken into account by the quantities

$$f_1(\mathbf{r}) = \frac{K_s(\mathbf{r}) - K_0}{K_s(\mathbf{r})} \qquad f_2(\mathbf{r}) = 2\frac{\rho_s(\mathbf{r}) - \rho_0}{2\rho_s(\mathbf{r}) + \rho_0} \tag{2}$$

where f_1 and f_2 are Gor'kov's parameters [2] for the bulk modulus K and density ρ of the object, also referred to as the scatterer (subscript s), relative to the corresponding properties of the surrounding fluid (subscript 0). Two applications of these generalizations are presented below.

First, the effect of material inhomogeneity is analyzed using closed-form expressions obtained from Eqs. (1) for the radiation force and torque exerted by a standing plane wave on a finite cylinder having material properties that vary linearly along its axis. The expressions reveal the dependence of the force and torque on the position and orientation of the object with respect to the incident field as a function of the degree of inhomogeneity.

Second, instead of a standing plane wave, a progressive spherical wave is considered for the incident field. The Born approximation relies on energy density gradients providing the dominant contribution to the radiation force and torque. [1] This is always the case in a standing wave, whereas in a progressive plane wave the dominant contribution is associated with the momentum in the scattered wave. Gor'kov [2] demonstrated that energy density gradients determine the radiation force on a small sphere located near the center of a diverging or converging spherical wave, and that scattering determines the force far from the center. Gor'kov's analysis is extended here to assessment of the Born approximation applied to a homogeneous compressible sphere of finite size in a diverging or converging spherical wave field. The assessment is based on comparison of a closed-form expression for the radiation force obtained from Eqs. (1) and the full solution based on spherical harmonic expansions of the incident and scattered fields [3, 4].

2 PLANE STANDING WAVE INCIDENT ON AN INHOMOGENEOUS CYLINDER

The standing plane wave acting on the inhomogeneous cylinder is described by the pressure field

$$p = p_0 \cos[k(z+d)] \cos \omega t \tag{3}$$

where $k = \omega/c_0$, c_0 is the speed of sound in the fluid surrounding the object, and d is a parameter introduced to shift the standing wave relative to the origin. Equations (1) for the force and torque become

$$F_z = \frac{p_0^2 k}{4\rho_0 c_0^2} \int_V f_G(\mathbf{r}) \sin[2k(z+d)]\, dV \qquad \boldsymbol{\tau} = \frac{p_0^2 k}{4\rho_0 c_0^2} \int_V (\mathbf{r} \times \mathbf{e}_z) f_G(\mathbf{r}) \sin[2k(z+d)]\, dV \tag{4}$$

where $\mathbf{F} = F_z \mathbf{e}_z$ and $f_G(\mathbf{r}) = f_1(\mathbf{r}) + \frac{3}{2} f_2(\mathbf{r})$.

The cylinder has length L and diameter D, its axis of symmetry is designated as z', which is assumed to lie in the x-z plane and form an angle θ with the z axis (with increasing θ corresponding to positive rotation about the y axis in accordance with the right-hand rule), and the center of the cylinder is positioned at the origin, $(x,y,z) = (0,0,0)$. The material inhomogeneity is assumed to vary linearly along the axis of the cylinder according to $f_G(z') = f_m + (2z'/L) f_\delta$, which increases with respect to the mean value f_m from $f_m - f_\delta$ at one end ($z' = -L/2$) to $f_m + f_\delta$ at the other ($z' = L/2$). Because the geometry and material properties of the cylinder are symmetric about the plane $y = 0$ the torque has only a y component, such that $\boldsymbol{\tau} = \tau_y \mathbf{e}_y$. Evaluation of the integrals in Eqs. (4) yields

$$F_z(d,\theta) = \frac{p_0^2 k V}{4\rho_0 c_0^2} \frac{2J_1(kD\sin\theta)}{kD\sin\theta} [f_m j_0(kL\cos\theta) \sin 2kd + f_\delta j_1(kL\cos\theta)\cos 2kd] \tag{5}$$

$$\tau_y(d,\theta) = \frac{p_0^2 V}{192\rho_0 c_0^2} [f_m \sigma_m(\theta) \cos 2kd - f_\delta \sigma_\delta(\theta) \sin 2kd] \tag{6}$$

where

$$\sigma_m(\theta) = \sin 2\theta \left[3k^2 D^2 \frac{8J_2(kD\sin\theta)}{(kD\sin\theta)^2} j_0(kL\cos\theta) - 4k^2 L^2 \frac{2J_1(kD\sin\theta)}{kD\sin\theta} \frac{3j_1(kL\cos\theta)}{kL\cos\theta}\right] \tag{7}$$

$$\sigma_\delta(\theta) = k^3 D^2 L \sin 2\theta \cos\theta \frac{8J_2(kD\sin\theta)}{(kD\sin\theta)^2} \frac{3j_1(kL\cos\theta)}{kL\cos\theta} + 8kL\sin\theta \frac{2J_1(kD\sin\theta)}{kD\sin\theta} [j_0(kL\cos\theta) - 2j_2(kL\cos\theta)] \tag{8}$$

and $V = \frac{1}{4}\pi D^2 L$ is the volume of the cylinder, J_n the cylindrical Bessel function, and j_n the spherical Bessel function. For $f_\delta = 0$ the results obtained previously [1] for a homogeneous cylinder are recovered.

The effects of inhomogeneity are illustrated most clearly in the limiting case of a small ($kD, kL \ll 1$) thin ($D^2 \ll L^2$) rod, for which Eqs. (5) and (6) reduce at leading order to

$$F_z(d,\theta) = \frac{p_0^2 kV}{12\rho_0 c_0^2}\big(3f_m \sin 2kd + kLf_\delta \cos 2kd \cos\theta\big) \tag{9}$$

$$\tau_y(d,\theta) = -\frac{p_0^2 kVL}{48\rho_0 c_0^2}\big(kLf_m \cos 2kd \sin 2\theta + 2f_\delta \sin 2kd \sin\theta\big) \tag{10}$$

In this limit it is observed that the effect of the inhomogeneity f_δ on the force F_z is smaller by a factor of order kL than the force associated with the mean value f_m of the material properties, whereas the reverse is true for the effect of the inhomogeneity on the torque τ_y. The force is affected significantly by the inhomogeneity only when the rod is located in the vicinity of a pressure node or antinode, corresponding to $kd \simeq n\pi/2$. In contrast, the torque is affected by inhomogeneity for all kd (except $kd \simeq n\pi/2$) provided f_δ/f_m is of order kL. The dependence of τ_y on $\sin\theta$ resulting from the inhomogeneity, as opposed to $\sin 2\theta$ for a homogeneous rod (corresponding to the term multiplied by f_m), is due to the inhomogeneity being an odd function of the axial coordinate with respect to the midpoint of the rod.

3 PROGRESSIVE SPHERICAL WAVE INCIDENT ON A HOMOGENEOUS SPHERE

The validity of the Born approximation for the radiation force on a homogeneous compressible sphere in a diverging or converging spherical wave field is assessed via comparison with the full solution based on spherical harmonic expansions. For a sphere in a plane standing wave field, as considered previously [1], energy density gradients provide the dominant contribution to the radiation force, as required by the Born approximation. In a progressive spherical wave field, energy density gradients dominate only near the geometric center of the field, whereas farther away the dominant contribution is associated with momentum in the scattered wave. Gor'kov [2] determined how close to the center of the spherical wave a small particle must be for energy density gradients to dominate. The purpose of the following analysis is to determine this distance for a compressible sphere of finite size, and thus determine the region in which the Born approximation is valid for a progressive spherical wave field.

3.1 Born approximation

Since the sphere is homogeneous (constant f_1 and f_2) there is no torque because of the symmetry of both the object and the incident field, and the expression for the force in Eqs. (1) becomes

$$\mathbf{F} = -\int_V \boldsymbol{\nabla}[f_1\langle E_p\rangle - \tfrac{3}{2}f_2\langle E_k\rangle]\,dV \tag{11}$$

The diverging or converging spherical wave incident on the sphere is centered along the z axis at $\mathbf{r} = \mathbf{r}_0 \equiv -r_0\,\mathbf{e}_z$:

$$p = \frac{1}{2}p_{\rm in}e^{-i\omega t} + \text{c.c.} \qquad p_{\rm in}(s) = p_0\frac{s_0}{s}e^{\pm iks} \tag{12}$$

where $s = |\mathbf{r} - \mathbf{r}_0|$ is distance from the center of the spherical wave, $|p_{\rm in}| = p_0$ at $s = s_0$, and the plus sign in $e^{\pm iks}$ corresponds to a diverging spherical wave, the minus sign to a converging spherical wave. The energy densities are the same in each case:

$$\langle E_p\rangle = \frac{p_0^2 s_0^2}{4\rho_0 c_0^2 s^2} \qquad \langle E_k\rangle = \frac{p_0^2 s_0^2}{4\rho_0 c_0^2 s^2}\left(1 + \frac{1}{k^2 s^2}\right) \tag{13}$$

Therefore the radiation force in the Born approximation is the same, both magnitude and direction, for both diverging and converging spherical waves.

The spherical wave is assumed to be incident on a spherical object of radius a that is centered along the z axis at $z=0$, such that the distance between the centers of the spherical wave and the spherical object is r_0. For simplicity it is further assumed here that $r_0 > a$ so that the center of the spherical wave is not located within the spherical object. In this configuration the net radiation force on the sphere has only a z component, and Eq. (11) yields

$$F_z = \frac{\pi p_0^2 s_0^2}{4\rho_0 c_0^2}\left[\left(f_1 - \frac{3}{2}f_2\right)g(\alpha) - \frac{3f_2}{k^2 r_0^2}h(\alpha)\right] \tag{14}$$

where

$$g(\alpha) = (1+\alpha^2)\ln\left(\frac{1+\alpha}{1-\alpha}\right) - 2\alpha \qquad h(\alpha) = \frac{\alpha+\alpha^3}{(1-\alpha^2)^2} - \frac{1}{2}\ln\left(\frac{1+\alpha}{1-\alpha}\right) \tag{15}$$

and $\alpha = a/r_0$ (< 1). When a/r_0 is sufficiently small, α is approximately the angle $\arcsin(a/r_0)$ subtended by the z axis and the perimeter of the sphere as viewed from $\mathbf{r} = \mathbf{r}_0$.

For $\alpha \ll 1$ both $g(\alpha)$ and $h(\alpha)$ reduce to $\frac{8}{3}\alpha^3 + O(\alpha^5)$ and Eq. (14) becomes

$$F_z = \frac{2\pi p_0^2 s_0^2}{3\rho_0 c_0^2}\left[f_1 - \frac{3}{2}f_2\left(1 + \frac{2}{k^2 r_0^2}\right)\right]\frac{a^3}{r_0^3} \qquad r_0 \gg a \tag{16}$$

Equation (16) is also obtained if Eq. (11) is replaced by $F_z = -(d/ds)[f_1\langle E_p\rangle - \frac{3}{2}f_2\langle E_k\rangle]\frac{4}{3}\pi a^3|_{s=r_0}$, which is consistent with Eqs. (12) and (15) of Gor'kov [2] for a small sphere in a progressive spherical wave field. In the opposite limit, $r_0 \to a$, the first term in $h(\alpha)$ determines the asymptotic behavior near the singularity at $r_0 = a$:

$$F_z \sim -\frac{3\pi p_0^2 s_0^2}{8\rho_0 c_0^2}\frac{f_2}{k^2(r_0 - a)^2} \qquad r_0 \to a \tag{17}$$

which depends only on the density contrast associated with f_2, and not on the compressibility contrast associated with f_1. This dependence is consistent with the nearly incompressible nature of a spherical wave field for small ks.

3.2 Transition points

If f_1 and f_2 are either both positive or both negative, as is typically the case, and if $f_1/f_2 > 3/2$, then F_z passes through a zero as the distance r_0 between the centers of the spherical wave and the spherical object is increased. From Eq. (14), this distance $r_0 = r_1$ is determined by the implicit relation

$$kr_1 = \left[\left(\frac{1}{3}\frac{f_1}{f_2} - \frac{1}{2}\right)\frac{g(ka/kr_1)}{h(ka/kr_1)}\right]^{-1/2} \qquad \frac{f_1}{f_2} > \frac{3}{2} \tag{18}$$

Note that $g/h \to 1$ as $a/r_1 \to 0$, and the explicit relation obtained from Eq. (16) for a small sphere, discussed previously by Gor'kov [2], is recovered.

Of particular interest is the transition point $r_0 = r_2$ where the contribution to the radiation force due to momentum transfer associated with scattering becomes comparable to that due to the energy density gradients in Eq. (11). The Born approximation for a sphere of finite size is valid only when energy density gradients provide the dominant contribution. An expression for the transition point r_2 in a spherical wave field is available only for small spheres, obtained by setting Eq. (16) equal to the result for a progressive plane wave incident on a small sphere [2]:

$$kr_2 = \frac{3|f_1 - \frac{3}{2}f_2|}{f_1^2 + f_1 f_2 + \frac{3}{4}f_2^2}\frac{1}{(ka)^3} \qquad ka \ll 1 \tag{19}$$

The corresponding value of r_2 for a sphere of finite size must be determined numerically by comparison with the full solution for the radiation force due to a spherical wave incident on the sphere, but Eq. (19) nevertheless provides a useful point of reference.

3.3 Comparison with the full solution for the force on a sphere

The full solution for the radiation force due to a spherical wave incident on the compressible sphere is [3, 4]

$$F_z = \frac{i\pi}{\rho_0 c_0^2 k^2} \sum_{n=0}^{\infty} \frac{(n+1)a_n^* a_{n+1}}{(2n+1)(2n+3)} (A_n^* + A_{n+1} + 2A_n^* A_{n+1}) + \text{c.c.} \tag{20}$$

where $a_n = i(-1)^n(2n+1)h_n^{(1)}(kr_0)ks_0p_0$ is the coefficient in the spherical harmonic expansion of the incident diverging spherical wave for $r_0 > a$ (with its complex conjugate used for the converging spherical wave), $h_n^{(1)}$ is the spherical Hankel function, and A_n is the coefficient in the expansion of the scattered wave.

Results for the radiation force are normalized by

$$F_0 = \frac{\pi a^2 p_0^2}{2\rho_0 c_0^2}\left(1 + \frac{1}{2k^2 r_0^2}\right)\frac{s_0^2}{r_0^2} \tag{21}$$

This reference force follows from the definition $F_0 = S_0\langle E_{\text{in}}\rangle$, where $S_0 = \pi a^2$ is the cross-sectional area of the sphere, and $\langle E_{\text{in}}\rangle$ is the total energy density $\langle E_p\rangle + \langle E_k\rangle$ in the incident field evaluated at $s = r_0$ (the center of the sphere). To maintain constant pressure amplitude $|p_{\text{in}}| = p_0$ at $s = r_0$ we set $s_0 = r_0$, such that for $kr_0 \gg 1$ the quantity F_z/F_0 is equivalent to the acoustic radiation force function Y_p traditionally employed for a progressive plane wave with pressure amplitude p_0 incident on a sphere of radius a.

Shown in Fig. 1 are results for $f_1 = 0.15$ and $f_2 = 0.05$ ($f_1/f_2 = 3$), which are nominal values for material properties corresponding to soft biological media in water, and for $ka = 0.4$ (first column), $ka = 0.7$ (second column), and $ka = 1$ (third column). The dashed curves are the full theory given by Eq. (20), with black dashes (upper curves) corresponding to diverging spherical waves, green dashes (lower curves) to converging spherical waves. The solid red curves are the Born approximation, Eq. (14), and the solid blue curves are Gor'kov's result for a small sphere, Eq. (16). The force is plotted on a linear scale in the first row of Fig. 1 and on a logarithmic scale in the second row, with the blue curves in the second row terminated at $kr_0 = ka$, showing only the range $r_0 > a$ encompassed by the present analysis.

For the smallest sphere considered, $ka = 0.4$, all four curves appear close together on the linear scale in Fig. 1(a), and for $ka \lesssim 0.2$ (not shown) they are virtually indistinguishable on the linear scale. However, differences become apparent on the corresponding logarithmic scale in Fig. 1(d). Below the zero crossing at $kr_1 = 1.5$ predicted by Eq. (18), i.e., in the region $kr_0 < kr_1$ where the radiation force pushes the sphere toward the center of the spherical wave, the Born approximation is in very close agreement with the full solution. In contrast, Gor'kov's result for a small sphere deviates from the full solution in this region. Above the zero crossing ($kr_0 > kr_1$), where the force initially pushes the sphere away from the center of the spherical wave, the Born approximation and Gor'kov's result become indistinguishable from one another, and both analytical results start to depart from the full solution at $kr_0 \sim 10$. The increasing discrepancy beyond $kr_0 \sim 10$ is consistent with the approach to the transition point $kr_2 = 110$ predicted by Eq. (19) where the energy density gradients no longer provide the dominant contribution to the radiation force, because the wave field increasingly resembles a progressive plane wave.

For the larger spheres, the aforementioned discrepancies with the full solution are increased. Here we discuss only the logarithmic plot for the largest sphere, $ka = 1$ in Fig. 1(f). Note that the Born approximation remains in good agreement with the full solution in the region $kr_0 < kr_1$, including at the location $kr_1 = 1.8$ of the zero crossing predicted by Eq. (18). Above the zero crossing ($kr_0 > kr_1$) the Born approximation and Gor'kov's result again become indistinguishable, but they immediately diverge from the full solution due to the proximity of the transition point $kr_2 = 7$ predicted by Eq. (19). For the intermediate sphere, $ka = 0.7$, the transition points are $kr_1 = 1.6$ and $kr_2 = 21$.

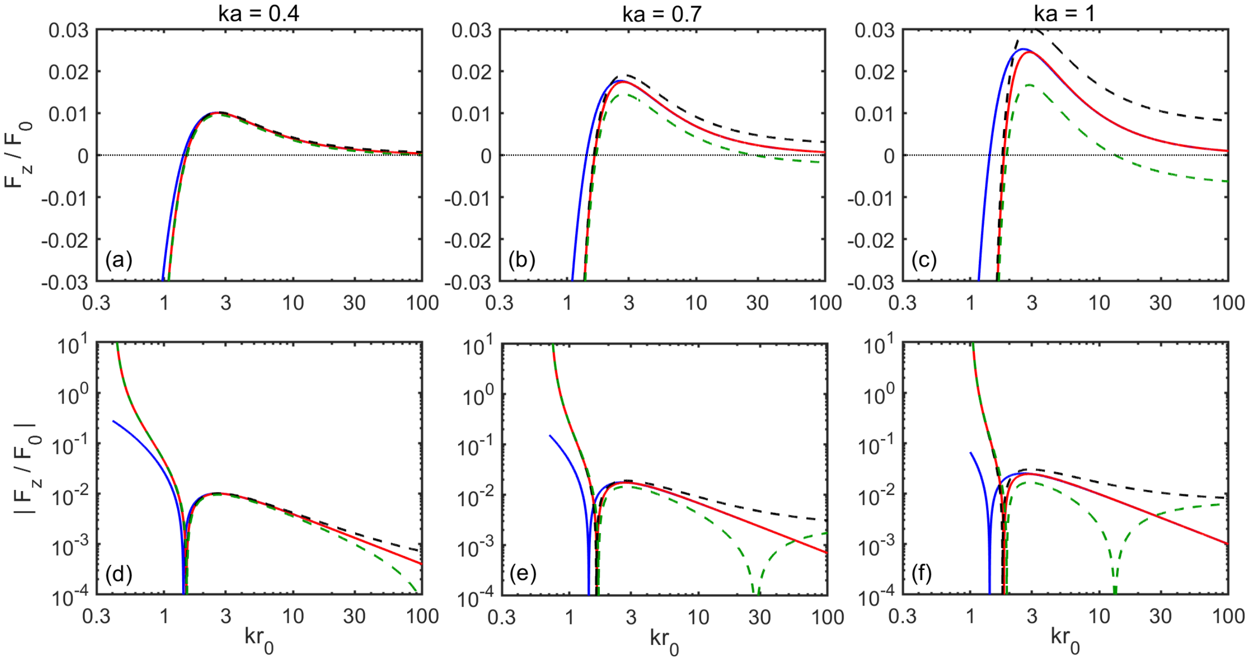

Figure 1. Comparisons of the acoustic radiation force F_z as a function of kr_0 for $f_1 = 0.15$ and $f_2 = 0.05$ obtained using Eq. (14) (Born approximation, solid red curves), Eq. (16) (Gor'kov's result for a small sphere, solid blue curves), and Eq. (20) (full solution, dashed black curves for diverging spherical wave, dashed green curves for converging spherical wave) for $ka = 0.4$ (first column), $ka = 0.7$ (second column), and $ka = 1$ (third column). The force is plotted on a linear scale in the first row, and on a logarithmic scale in the second row.

The corresponding curves for $(f_1, f_2) = (-0.15, -0.05)$, for which the values of kr_1 and kr_2 are the same as those for $(f_1, f_2) = (0.15, 0.05)$, are very close to the curves in the second row of Fig. 1, and they are virtually mirror images with respect to $F_z = 0$ of the curves in the first row. The main difference is that the curves for the diverging and converging waves based on the full solution are reversed because for sufficiently large kr_0, where the field resembles a progressive plane wave, the diverging wave must eventually push the sphere outward, and the converging wave must eventually push the sphere inward, regardless of the material properties. The analysis in the preceding two paragraphs is unaffected by this distinction.

3.4 Discussion

In summary, for f_1 and f_2 either both positive or both negative, the Born approximation of the radiation force on a soft compressible sphere in both diverging and converging spherical wave fields is in good agreement with the full solution in the region $a < r_0 < r_1$ for ka as large as $O(1)$. In the vicinity of its singularity at $r_0 = a$, the Born approximation agrees very well with the full solution. For $ka \lesssim 0.4$, the Born approximation is accurate beyond $r_0 = r_1$ out to $kr_0 \sim 10$.

While the case $r_0 < a$ was not considered here mainly to simplify the analysis, it corresponds to the situation where the center of the spherical wave is located inside the spherical object. In this case there is little physical sense to the assumption of purely progressive spherical waves, because a diverging wave would require a point source to be located within the object, and a purely converging wave cannot exist without disappearing upon passage through the focus inside the object. The case $r_0 < a$ is therefore most relevant to standing spherical waves, for which energy density gradients provide the dominant contribution to the radiation force everywhere in the field, and this restriction on the Born approximation is not in question.

The advantages of the Born approximation are that it applies to objects of any shape—subject to certain restrictions on material properties and size of the object—and that torque as well as force can be calculated rather easily. [1] Although the accuracy of the Born approximation for an object in a diverging or converging spherical wave field is assessed here only for a spherical object, the nominal domain of validity should apply to nonspherical objects of comparable size.

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REFERENCES

[1] Jerome, T.S.; Ilinskii, Yu.A.; Zabolotskaya, E.A.; Hamilton, M.F. Born approximation of acoustic radiation force and torque on soft objects of arbitrary shape. Journal of the Acoustical Society of America, Vol. 145, 2019, pp. 36–44.

[2] Gor'kov, L.P. On the forces acting on a small particle in an acoustical field in an ideal fluid. Soviet Physics–Doklady, Vol. 6, 1962, pp. 773–775.

[3] Ilinskii, Yu.A.; Zabolotskaya, E.A.; Treweek, B.C.; Hamilton, M.F. Acoustic radiation force on an elastic sphere in a soft elastic medium. Journal of the Acoustical Society of America, Vol. 144, 2018, pp. 568–576.

[4] Sapozhnikov, O.A.; Bailey, M.R. Radiation force of an arbitrary acoustic beam on an elastic sphere in a fluid. Journal of the Acoustical Society of America, Vol. 133, 2013, pp. 661–676.