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Applying the diffusion equation to urban scenarios: Computational analysis of the diffusion coefficient

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ABSTRACT

Prediction of sound propagation is playing a key role in the planning and design process of urban areas and during the last decades different techniques have been developed for the computation of the sound fields in cities. Among them, the diffusion equation, based on the propagation of sound energy, is a simple and attractive tool for certain scenarios where the diffuse field is predominant, such as inner city environments. The diffusion equation is a well-known and efficient method to compute sound fields in rooms with a low amount of absorption. However, this method, which includes the diffusion coefficient describing the diffusivity of the energy propagation, requires further understanding for urban environments. In this work, the diffusion coefficient has been evaluated using wave-based numerical techniques for simplified two-dimensional urban scenarios. The parameter is computed spatially from the acoustic intensity and sound energy density fields in every direction for different geometries. This paper investigates the directional components of the diffusion coefficient.

Keywords: Diffusion equation, urban sound propagation

1 INTRODUCTION

The diffusion equation describes sound propagation by a diffusion process of the sound energy. The method relies mainly on a single parameter - the diffusion coefficient D - that requires further understanding for urban environments. The hypothesis assumed in this paper is that the sound field can be approximated by a diffusion process in canyon-type street configurations. For these cases, the diffusivity of sound energy has different properties in different directions and as a consequence, the diffusion coefficient is not uniform (1). Although some authors have derived analytical expressions or methodologies to estimate the coefficients for urban cases, for instance (1) and (2), there is still a lack of knowledge regarding how the diffuse sound energy propagates in these cases. The main contribution of this paper is to investigate the directional components of the diffusion coefficient by using a novel methodology to spatially compute D using solutions of a wave-based method for simplified urban environments. Section 2 presents the principles of the diffusion equation for sound propagation with special emphasis on the presentation of the non-homogeneous and directional diffusion coefficient for urban cases. In this work, the computation of the coefficients is done by using the sound pressure and acoustic velocity as calculated by the discontinuous Galerkin (DG) wave-based time domain method; the approach is also valid for any numerical method that allows the steady state acoustic intensity and sound energy density to be computed at discrete positions of the domain under investigation. The full methodology of obtaining the diffusion coefficients is detailed in Section 3. The approach has been used in two simplified two-dimensional scenarios, described in Section 4: a straight, long street and an intersection of two perpendicular long streets. The solutions of the diffusion coefficients for these scenarios are presented in Section 5 and the final conclusions can be found in Section 6.

2 THE ACOUSTIC DIFFUSION EQUATION

Fick's first law, adapted to a diffuse sound energy field, is presented in Equation 1, where $w(r,t)$ $\left[\mathrm{kg\,m^{-1}\,s^{-2}}\right]$ is the sound energy density. This law states that the flow of acoustic energy (sound intensity vector J $\left[\mathrm{kg\,s^{-3}}\right]$) occurs from regions of high energy to areas of low energy concentration, with a proportionality relation between the acoustic intensity and the sound energy density gradient (∇w). The proportionality factor in the equation is the diffusion coefficient D $\left[\mathrm{m^2\,s^{-1}}\right]$. The derivation of Fick's second law - or more popularly the diffusion equation (DE) - for diffuse sound fields was presented by (4) (see Equation 2). The absorption by the domain surfaces is included in σ $\left[\mathrm{s^{-1}}\right]$, which represents the probability rate of a sound particle to be absorbed during one second. The DE presented in Equation 2 represents sound energy density propagation in the time domain. In order to obtain frequency solutions, the equation needs to be resolved for each frequency band of interest separately.

$$J(r,t) = -D\nabla w(r,t). \tag{1}$$

$$\frac{\partial}{\partial t} w(r,t) = D\nabla^2 w(r,t) - \sigma w(r,t). \tag{2}$$

The diffusion coefficient is defined for room acoustics problems by Valeau et al. in (3) as a term of the diffusion equation that takes into account the room morphology through the mean free path λ [m]. This classical definition corresponds with the theoretical derivations presented in (4) for rooms of proportionate dimensions, where D takes a constant value depending on the geometrical properties of the space as $D = \lambda c_0/3$, where c_0 is the sound velocity and $\lambda = 4V/S$, with V and S the volume and surface area of the proportionate room. However, when the space is more complex, as for instance in a long room or in a canyon-type street, the diffusion coefficient is not as simple. Visentin et al. (5) carried out an investigation to study the validity of Fick's law of diffusion in room acoustics from a numerical evaluation of the gradient Equation 1. One of the main conclusions of the study was that $D = \lambda c_0/3$, is only valid in areas close to the source in long rooms, while linearly increasing with distance from the source. Therefore, they proved that the sound field in long rooms is described by non-homogeneous diffusion, i.e., with a spatially varying diffusion coefficient. In this type of space, where one dimension is much bigger than the others, the mean free path is not homogeneous and spatially depends on the location of the source and receiver. In receiver areas close to the source in an infinite long room, the distance between successive reflections is shorter (limited by the distance between parallel surfaces when the sound waves travel as a standing wave in the corridor), than in areas far from the excitation, where the sound waves have on average a larger angle with respect to the source (assuming zero degrees angle is perpendicular to the side walls). The hypothesis assumed in the present paper is that the sound field can be approximated by a diffusion process in canyon-type streets configurations by using the DE model as presented in Equation 2 without fundamental modifications. In this type of street, the diffusivity of sound energy has different properties in different directions and, as a consequence, the diffusion coefficient is not uniform and depends on the direction of propagation (1). The spatially dependent directional diffusion coefficients are presented in Equation 3 for three-dimensional rectangular streets in Cartesian coordinates. The diffusion gradient Equation 2, assuming the diffusion coefficients $D(r)$ given in 3, is employed to calculate the spatially dependent directional diffusion coefficients of urban cases.

$$\mathbf{D}(\mathbf{r}) = \begin{pmatrix} D_x(\mathbf{r}) & 0 & 0 \\ 0 & D_y(\mathbf{r}) & 0 \\ 0 & 0 & D_z(\mathbf{r}) \end{pmatrix}. \tag{3}$$

3 METHOD TO OBTAIN THE DIFFUSION COEFFICIENT FOR URBAN APPLICATIONS

In this Section, the methodology for obtaining spatially dependent directional diffusion coefficients $D(r)$ is presented. The approach begins with the frequency domain $J(f)$ and $\nabla w(f)$[1], calculated from solutions of a full wave-based method at a large number of equally-spaced discrete locations in the physical domain under investigation. The method is detailed for two-dimensional rectangular cases. The approach is based on solving Equation 1 by evaluating the ratio between the solutions of the steady state components of J and the corresponding energy density gradients ∇w.

One assumption in the derivation of the diffusion Equation 2 is that the variations per mean free path of w, J, and thus D, must be small. However, the solutions of a full wave-based method, as proposed here, implicitly include the interference effects due to the wave nature of sound propagation and therefore, the results are not as smooth as the diffusion equation would demand. In order to reduce this effect, calculating the energy density and the intensity in broader frequency bands, as indicated in Section 3.1, contributes to smoothing the results, as well as the local spatial averaging, performed according to Section 3.2. After the calculation and local averaging, the sound energy density gradient is computed as presented in Section 3.3. In Section 3.4, the gradient and the acoustic intensity are then averaged over the narrow dimension of the street. Before obtaining the diffusion coefficients, a smoothing process over the acoustics intensity and sound energy density is applied according to Section 3.5. Finally, the diffusion coefficients are obtained according to Section 3.6. Additionally, the variations per mean free path of the energy density and the acoustic intensity are not expected to be smooth along the narrow dimension of a street canyon. Therefore, the methodology presented in this section is only employed for computing the diffusion coefficients along the direction of the biggest dimension of the street.

3.1 Broad band frequency calculation

The energy density and the active part of the acoustic intensity, indicated generically in Equation 4 by $\Psi(r,f)$, are calculated for a broad frequency band f_{bb} by a summation of the results from the lowest frequency of interest f_l to the highest f_u, as indicated in Equation 4.

$$\Psi(r,f_{bb}) = \sum_{f=f_l}^{f_u} \Psi(r,f). \tag{4}$$

3.2 Receivers and local spatial averaging process

A large number of discrete locations r in the spatial domain is needed in order to obtain a meaningful representation of the sound field to get the diffusion coefficients. The total number of receivers in the whole domain is N_r and the distance between receivers is denoted as Δr [m] for every Cartesian dimension. Δr is chosen to be constant everywhere in the domain for the ease of the averaging process and the later calculation of the spatial energy gradient. The spatial average of the broadband w and J are computed at a number of equally spaced recording positions in each Cartesian direction n_r (same number in each direction). They are arithmetically averaged and the result is assigned to the central position. The total number of averaged positions is $n_{\bar{r}} = n_r^2$. The total number of evaluation positions N_r is reduced to $N_{\bar{r}} = \lfloor N_r/n_{\bar{r}} \rfloor$, and the distance between positions is now $\Delta\bar{r}$. The averaged frequency solutions are referred as $J_x(\bar{r})$, $J_y(\bar{r})$ and $w(\bar{r})$ for a certain frequency range.

3.3 Energy density gradient calculation

The next step is to compute the energy density gradient over the whole domain by using central differences for interior data points, while using single-sided differences along the edges of the domain. The following notation is used for the spatial discretisation of the energy density in the two-dimensional domain: $w_{i,j} = w(i\Delta\bar{r}, j\Delta\bar{r})$. The gradient

[1] Unless otherwise indicated, the reference to the frequency domain in the solutions has been omitted for subsequent references to these quantities, i.e. $J(f) = J$ and $\nabla w(f) = \nabla w$.

calculation in the x-direction is described in the following Equations 5 for interior data points and 6 for the positions located at the edges (equivalent expressions are used for the y-direction), where $N_{\bar{x}}$ is the total number of averaged points in the x-direction:

$$\nabla_x w_{i,j} \approx \frac{w_{i+1,j} - w_{i-1,j}}{2\Delta\bar{r}}, \tag{5}$$

$$\nabla_x w_{1,j} \approx \frac{w_{2,j} - w_{1,j}}{\Delta\bar{r}}, \quad \nabla_x w_{N_{\bar{x}},j} \approx \frac{w_{N_{\bar{x}},j} - w_{N_{\bar{x}}-1,j}}{\Delta\bar{r}}. \tag{6}$$

3.4 Averaging from 2D to 1D solutions

Since the streets considered in this approach are rather narrow, a second spatial averaging process is performed over the results $J_x(\bar{r})$, $J_y(\bar{r})$, $\nabla_x w(\bar{r})$ and $\nabla_y w(\bar{r})$. Essentially, the two-dimensional solutions are transformed into a set of one-dimensional results for each Cartesian dimension by spatially averaging the results over the other direction. In this approach, it is assumed that the solutions are constant along the narrower dimension of the street. When one of the results is expressed, for instance, by $J_x(\bar{r}_x)$, it means that the values have been averaged and only spatial values along the x-dimension remain, i.e. the values in the y-direction have been averaged.

3.5 Smoothing process over the sound energy density and the acoustic intensity

Despite the spatial averaging process and the calculation of solutions in a broad frequency band, the results still present strong variations within the spatial domain. In order to reduce these local variations, a moving averaged process is applied to the solutions obtained in the previous Section 3.4, that is equivalent to low-pass filtering the data. The response of the smoothing process is given by the Equation 7 for a generic solution $\psi_x(\bar{r}_x)$. The filter smooths the data by replacing each data point with the average of the neighbouring data points defined within a certain span defined as $n_{span} = 2N_{sm} + 1$.

$$\psi_{x,sm}(i) = \frac{1}{(2N_{sm}+1)}[\psi_x(i+N_{sm}) + \psi_x(i+N_{sm}-1) + \cdots + \psi_x(i-N_{sm})], \tag{7}$$

where $\psi_{x,sm}(i)$ is the smoothed value for the i^{th} data point and N_{sm} is the number of data points at each side of point i. The span must be an odd integer number. Moreover, the end points of the function are not smoothed. Finally, the span is adjusted for data points that cannot accommodate the specified number of neighbours on either side by reducing the number of points.

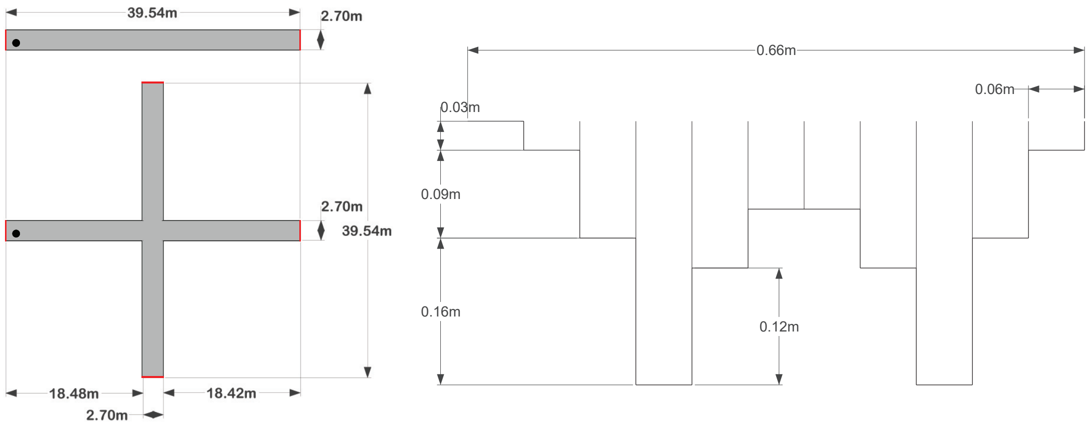
3.6 Calculation of the diffusion coefficient

The diffusion coefficients are finally computed at the averaged positions, for the frequency range of interest, using Equation 8 for the *x-dimension* and an equivalent expression for the *y-dimension*.

$$D_x(\bar{r}_x) = -\frac{J_x(\bar{r}_x)}{\nabla_x W(\bar{r}_x)}, \tag{8}$$

4 WAVE-BASED URBAN SCENARIOS

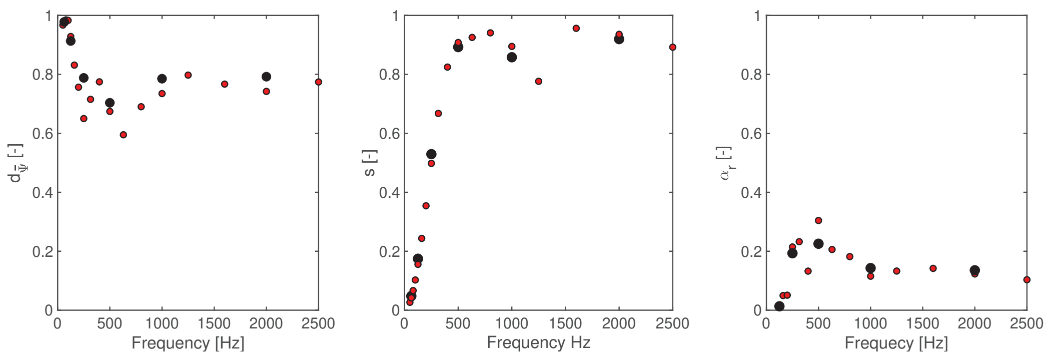
Two simplified urban problems have been investigated in this paper: a straight, long street (LS) and the perpendicular intersection of two long streets (LX). The two-dimensional domains are shown in Figure 1a. The façades of the scenarios are built using quadratic residue diffusers (QRDs) in order to obtain high values of the surface diffusion (d_Ψ) and scattering (s) coefficients. The arrangement of the QRDs followed the indications in (7) and the design has been done for frequencies higher than 500 Hz. The geometrical details of the QRD are given in Figure 1b. The DG scenarios have been designed for a frequency range of interest from 500 to 2500 Hz. The QRDs represent façade irregularities with a constant low sound absorption coefficient, while the beginning and the end of the streets are simulated as completely absorbent openings. The calculations in DG are initiated with a broadband pressure distribution and zero acoustic velocity.



(a) (b)

Figure 1. a) Dimensions of the two-dimensional urban domains including the source position (black dot) and the openings (red lines): upper domain straight, long street (LS), lower domain perpendicular intersection of two long streets (LX). b) Geometrical detail of the two-dimensional QRD used at the boundaries of the scenarios to simulate the façade's irregularities.

The results of the absorption coefficient for random sound incidence α_r, and the scattering and the diffusion coefficients (averaged value for all source positions $d_{\overline{\Psi}}$) according to (7) for the used QRD diffuser are presented in Figure 2. These values were obtained from a numerical evaluation using the DG model.



(a) (b) (c)

Figure 2. Results of the QRD used in this study for 1/1 (black big circles) and 1/3 (red small circles) octave bands: a) diffusion coefficient $d_{\overline{\Psi}}$, b) scattering coefficient s, and c) random absorption coefficient α_r.

4.1 Removing the direct sound and specular reflections

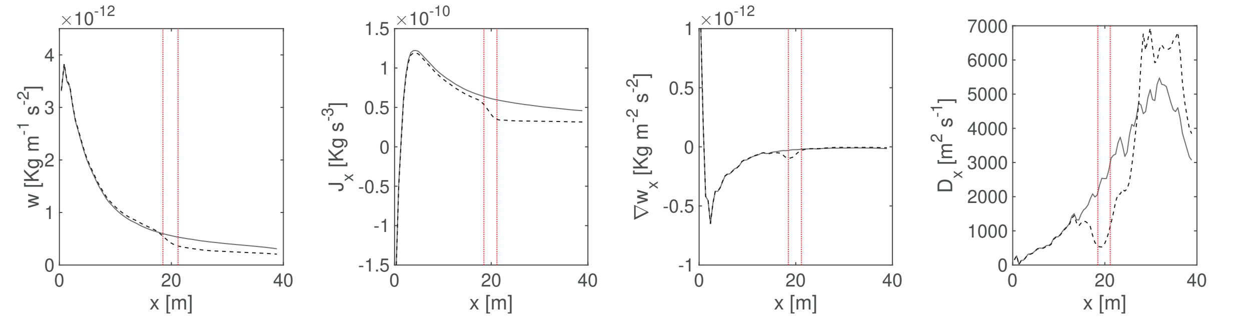
The nodal discontinuous Galerkin (DG) numerical time-domain wave-based method (6) was employed in this work for solving the linearised Euler equations representing sound propagation in urban scenarios. The impulse responses (IRs) of the acoustic solutions of DG are recorded at equally spaced multiple locations r in the domain. These solutions represent the steady state condition when transformed to the frequency domain. The source and receivers locations in the two-dimensional coordinate system are indicated as $r_s = [x_s, y_s]$ and $r_{n_r} = [x_{n_r}, y_{n_r}]$, respectively. The time step is computed depending on the maximum frequency of interest f_{max} according to $\Delta t < 1/(2.5 f_{max})$ $[s]$. The end of the time signals is tapered by a single-sided Gaussian window with a length ≈ 6 ms to avoid Gibbs effects. As by definition, the diffusion equation does not include the coherent part of the sound field, the beginning of the IRs is tapered by windowing the first part of the IRs. The window length is constructed by computing the number of time steps (n_{1a}) the direct sound takes to arrive to every receiver location in the case of receivers located in the line-of-sight from the source. The cases where the receiver location is not in the line-of sight from the source, the number of time steps until the first arrival (n_{1a}) has been computed from the distance between the source and the closest diffracted corner, plus the distance from that corner to the receiver position. The windowing function Γ_w is constructed using Equation 9, where the total number of recorded time steps is denoted by N_t with indexes $n_t = (1, 2, \ldots, N_t)$ and $\alpha_w = 7$ and $\beta_w = 4$ are constants controlling the shape of the exponential function. The length of the exponential window N_w is fixed in this work ($N_w \Delta t \approx 2.4$ ms) to remove the direct sound and assuming that all reflections are non-coherent for frequencies above 500 Hz, according to the design of the QRD.

$$\Gamma_w = \begin{cases} 0 & \text{for } 1 \leq n_t \leq n_{1a}, \\ e^{-\alpha_w log(10)\left(\frac{n_t - N_w}{N_w}\right)^{2\beta_w}} & \text{for } n_{1a} < n_t \leq n_{1a} + N_w, \\ 1 & \text{for } n_{1a} + N_w < n_t \leq N_t. \end{cases} \tag{9}$$

5 RESULTS OF THE WB SCENARIOS AND CALCULATION OF D

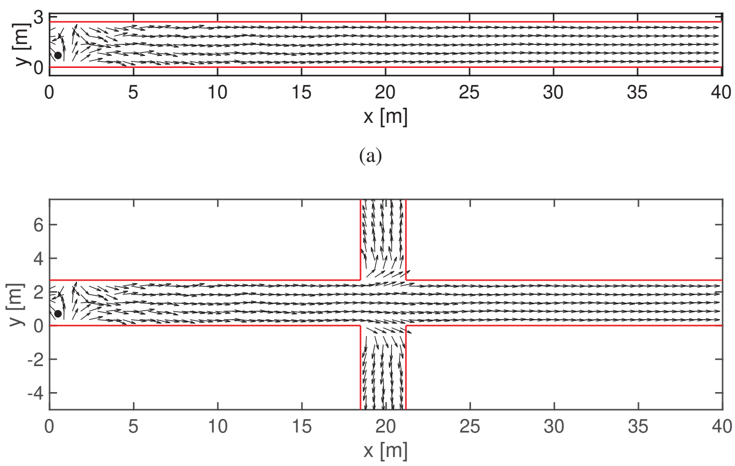
In this section, the results of the acoustic intensity, the energy density and the gradient of the energy density are presented as calculated from the solutions of the WB method. Additionally, the calculated diffusion coefficients are shown for the computed cases, LS and LX. The set of graphs shown in Figure 3 presents the results of w, J_x, $\nabla_x w$ and D_x of both scenarios. Excluded from this initial analysis the areas in the proximity of the source and street openings. The one-dimensional solutions of the LS case in Figure 3 show an increase of the diffusion coefficient along the street, while the energy density resembles an exponential decay as the distance from the source increases. This is in line with the diffusion coefficients reported by Visentin (5) for long rooms. As a consequence of the decay of the energy density, the gradient smoothly approaches zero as the slope of the energy decay decreases (see Figures 3a and 3c). The intensity follows a similar exponential pattern as the energy density, smoothly decaying with distance from the source as shown in 3b. Close to the right end of the main streets, there is a predominant direction of the intensity, as shown in Figure 4. However, in this area there is a drop of the value of the diffusion coefficients that can be explained due to the fact that the back-scattering in nearby areas of the openings is highly reduced. Moreover, the intensity field around the source location is not presenting a predominant direction, as shown in Figure 4, that can be additionally observed by the change of sign of J_x in Figure 3b. In the same area, the energy density presents its maximum value, decaying towards both openings (see the area around the source location in Figure 3a). Overall, it can be concluded that the openings of the streets affect the results in their proximity, questioning the validity of the methodology for computing the diffusion coefficients in these areas.

In the LX scenario, the energy density is distributed along the main street, but part of the energy is distributed to the side streets as shown in Figure 4b. The flow of energy occurs from the source towards the ends of the main and side streets. In this scenario, the results of the acoustic variables (w, J, ∇w and D) are presented in Figures 3 and 5, for the main and side streets, respectively. Each graph includes red dotted vertical lines indicating the position of the crossing street. Of particular interest in the main horizontal street is the analysis of the intensity field direction around the crossing area that can be observed in Figure 4b. The energy density presented in Figure 3a for this scenario is



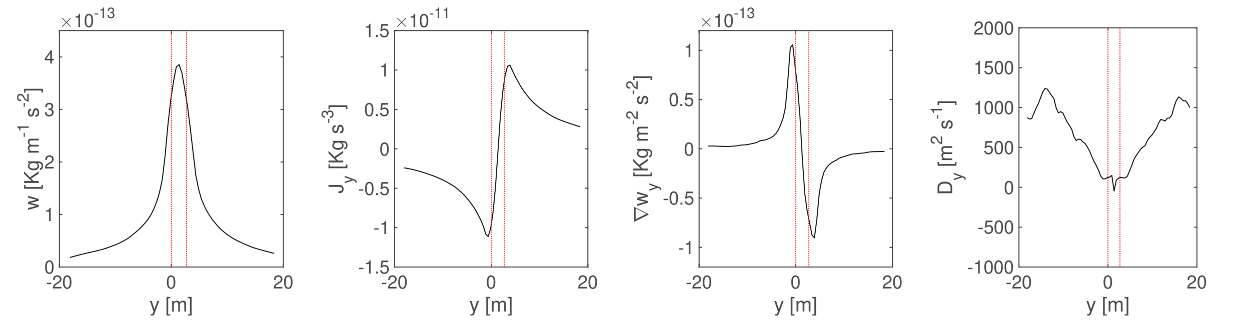
(a) (b) (c) (d)

Figure 3. Results of the LS scenario (grey solid line) and the main street of the LX scenario (black broken line). a) w, b) J_x, c) $\nabla_x w$ and d) D_x. The red dotted vertical lines represent the position of the crossing street (only for the LX scenario).



(b)

Figure 4. Direction of the acoustic intensity field of the street scenarios: a) LS and b) LX.



(a) (b) (c) (d)

Figure 5. Results of the two-dimensional cross street of the intersection model (LX). a) w, b) J_x, c) $\nabla_x w$ and d) D_x.

smoothly decaying until the crossing area where a more significant drop occurs, corresponding with the distribution of the energy towards the side streets. This can be observed as well in the results of J_x (see Figure 3b). Clearly, after the crossing area, the energy density and the acoustic intensity decay at a slower rate than before the crossing. The diffusion coefficient results present a clear drop in the main street right before the crossing and a sudden increase after the crossing. In the cross street, the analysis is equivalent. The energy density is distributed from the crossing to the side streets. Figure 4b shows how the acoustic intensity propagates with opposite directions in each side street and Figure 5b shows how the intensity distribution is approximately mirror-symmetric. The coefficients present an increase with distance when entering the side streets until the proximity of the openings, as shown in Figure 5d.

6 CONCLUSION

A novel methodology for the calculation of diffusion coefficients for the diffusion equation has been proposed for urban applications, based on the solutions of a wave-based method. The approach is based on locally computing the diffusion coefficients as the proportionality factor between the acoustic intensity and the gradient of the sound energy density given by Fick's first law of diffusion. The variations per mean free path of w, J, and thus D, must be small and therefore, efforts of the proposed methodology are oriented towards reducing the fluctuations of the solutions when using a full WB model. The presented approach has been employed for two-dimensional[2] scenarios composed of rectangular domains where one dimension is much bigger than the other. The method has been applied to a straight, long street and to a more complex urban case, the intersection of two perpendicular long streets. The proposed methodology to compute the diffusion coefficients has been proven to be a valid approach in areas where the sound field is smoothly decaying in a predominant direction. However, the method fails when the gradient of the energy density approaches zero. These cases are associated with areas of the domain where there is not a predominant and unique direction of the energy flow as near the source, openings or the narrow direction of the streets. The diffusion coefficients are found to be non-homogeneous, equivalently to the findings for long room cases (5).

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[2] Due to the maximum frequency of interest and the dimensions of the domain, it is not possible to employ the proposed technique using a three-dimensional WB model. This issue remains as a challenge for future work.