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Direct and inverse Hopf bifurcation in a neutral delay differential equation model of reed conical instrument

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Abstract

In conical musical instrument, the self-sustained oscillations appear at a blowing pressure threshold. At this threshold, the mathematical model encounters a so-called Hopf bifurcation. The bifurcation can be either direct or inverse which determines the stability of the arising oscillation. It is well known that for cylindrical instruments (clarinet-like instruments), the bifurcation is direct most of the time. In the case of conical instruments, the nature of the bifurcation has been studied analytically before, for instance using geometrical approximations based on cylinders. It is expected to find an inverse bifurcation in some cases. The proposed study addresses the bifurcation of an idealized model of reed instrument with a lossless conical resonator and a compliant mouthpiece, using two methods : a direct time-integration algorithm and a Taylor series based continuation method of periodic solutions, based on the harmonic balance method. It allows to draw the whole branch of periodic solutions and to deduce the type of Hopf bifurcation from a full diagram. Super-critical Hopf bifurcations can be found, with two fold bifurcations whose positions depend on the value of the geometrical parameters of the instrument.

Keywords: Hopf bifurcation, conical instrument, neutral delayed differential equations, harmonic balance method, time-integration

1 INTRODUCTION

The study of periodic solutions of physical models has shown to be an efficient tool to understand the behavior of the systems considered. A first common application is to predict and avoid undesired instabilities. In musical acoustics, the periodic solutions are sought for as they correspond to the regimes where an instrument produces a note. A lot of methods are available in the literature to study these regimes. Among those, two essentially different approaches will be considered. The first one is time-integration method. From the knowledge of the system before the starting time of integration, it reconstructs the behavior of the system for all greater times. Without *a priori* knowledge on the solution type, it is then possible to reconstruct the whole signal or waveform. For all the values of the parameters, a new time-integration is performed and thus the behavior of the system is known on a range of the parameters. However, this can be more complex in the case of several co-existing solutions and these methods are often unable to obtain unstable regimes. If the aim is the study of periodic solutions, the transient regime before the steady state can be very long and difficult to characterize. The second type of approach is continuation methods. These methods allow to keep track of the waveform as one parameter of the system is varied. Only steady-state solutions are considered. Thus, from one starting periodic solution a whole branch of periodic solution is determined. From the solution branch, it is possible to determine the critical value of the parameter above which a sound arise (the Hopf bifurcation point). A study of the oscillation thresholds of the clarinet by continuation is proposed in [8].

In this short article, the two complementary approaches are applied to a simplified model of saxophone. Previous work treat the bifurcation of saxophone-like instruments using multi-cylindrical approximations of the resonator [5, 6] or approaches based on the input impedance [7]. The model we use here has a conical resonator modelled by a delayed system. It is taken from [3]. Although losses and reed dynamics are ignored, this model has been shown to reproduce internal pressure waveform for the saxophone. A reformulation of the model leads to a strongly nonlinear neutral delay differential equation, for which no analytical solution exists.

However, the equation can be solved numerically using sound synthesis (time-integration) or harmonic balance. From sound synthesis, the general features of the produced sound are determined in a two-dimensional space of parameters value. This study helps to give a hint of the behavior of the system but fails to state clearly whether the Hopf bifurcation is super-critical of sub-critical. In practice, this information is interesting from the musician point of view. The super-critical case features a soft appearance of the sound in the instrument while the sub-critical case is often linked with a sudden appearance of the sound and a hysteresis loop. The continuation study of the same model is performed with a Taylor series based method coupled with the harmonic balance method [2]. It allows to have a continuous representation of the solution-branch. From the continuation diagram, the nature (super- or sub-critical) of the Hopf bifurcation and the hysteresis loops can be determined. The results obtained with the two methods are compared to validate the approaches.

2 Model and numerical methods

2.1 Model

The acoustical variables at the input of the resonator p and u are in dimensionless form, obtained from the the physical values $\hat{p}$ and $\hat{u}$ as

$$p = \frac{\hat{p}}{p_M} \quad , \quad u = Z_c \frac{\hat{u}}{p_M}, \tag{1}$$

where p_M is the static pressure necessary to close the reed channel completely, and $Z_c = \frac{\rho c}{S}$ is the characteristic impedance. The control parameters of the model are

$$\gamma = \frac{p_m}{p_M} \quad , \quad \zeta = Z_c w H \sqrt{\frac{2}{\rho p_M}}, \tag{2}$$

where p_m is the pressure in the mouth of the musician, w is the width of the reed channel and H is the height of the reed channel at rest. The control parameter ζ represents dimensionless reed opening at rest.

The reed is considered as a spring without mass: its position is proportional to the pressure difference between the input of the instrument and the mouth of the musician. The nonlinear characteristic giving the flow u depending on the pressure p and the control parameters is deduced from the Bernoulli law [4] and writes

$$u = F(p) = \begin{cases} \zeta(1-\gamma+p)\mathrm{sign}(\gamma-p)\sqrt{|\gamma-p|} & \text{if} \quad p \geq \gamma-1 \\ 0 & \text{if} \quad p < \gamma-1 \end{cases} . \tag{3}$$

The resonator of the saxophone is considered as a lossless conical waveguide with a lumped mouthpiece, whose volume equals the volume of the missing apex of the cone [3]. The pressure p at the input obeys the equation

$$\frac{dp}{dt}(t) = \frac{dp}{dt}(t-\hat{\tau}) - \frac{6c}{x_1}\left[p^-(t) + p^+(t-\hat{\tau}) + \frac{c}{2x_1}\int_{t-\hat{\tau}}^{t} p(t')dt'\right], \tag{4}$$

where $p^+ = (p+u)/2$ and $p^- = (p-u)/2$ and the parameters are x_1 the length of the missing apex of the cone, $\hat{\tau} = 2\ell/c$ is the duration of a round trip in the resonator of length ℓ.

Deriving equation (4) leads to an equation in p only, that can be used for harmonic balance:

$$\ddot{p} + \sqrt{3}(1-F'(p))\dot{p} + p = \ddot{p}_\tau - \sqrt{3}(1+F'(p_\tau))\dot{p}_\tau + p_\tau, \tag{5}$$

where p_τ is the delayed pressure and the time and delay are dimensionless:

$$\tilde{t} = \frac{c}{x_1}\sqrt{3}t \quad , \quad \tau = \frac{2\ell}{x_1}\sqrt{3}. \tag{6}$$

The dot notation signals derivation with respect to the dimensionless time. With the dimensionless time, there are only 3 parameters : the control parameters ζ and γ, and the delay τ which is proportional to the ratio ℓ/x_1. In this paper, ζ is fixed at 1 and x_1 at 0.4 m, which is a typical value for alto saxophones, as we concentrate on the effects of parameters γ and ℓ.

2.2 Sound synthesis

Equation (4) may be discretized to serve for sound synthesis synthesis with time step t_s:

$$I_n = I_{n-1} + \frac{ct_s}{4x_1}\left(p_n + p_{n-1} - p_{n-2M} - p_{n-2M-1}\right) \tag{7}$$

$$p_{n+1} = p_n + p_{n-2M+1} - p_{n-2M} - \frac{6ct_s}{x_1}\left(\frac{1}{2}(p_n - u_n) + \frac{1}{2}(p_{n-2M} + u_{n-2M}) + I_n\right) \tag{8}$$

$$u_{n+1} = F(p_{n+1}), \tag{9}$$

where $M = \ell/(ct_s)$ and I_n is the discretized version of the integral of p.

In order to explore the oscillating regimes of the model, a cartography is calculated by time-integration, describing the peak-to-peak amplitude of the last 0.1 s of the pressure signal p in simulations of 1 second of sound each, with different values of γ and ℓ. Figure 1 presents this cartography. The mouth pressure parameter γ varies from 0.25 to 0.5, around the threshold where oscillations appear, at about 1/3. The lengths ℓ range from 0.3 m, below x_1, to 1.1 m which is approximately the total length of an alto saxophone resonator. A change in behavior appears at the apparition of the oscillation ($\gamma \simeq 0.35$), depending on the value of the length ℓ: for $\ell < x_1$ the amplitude of the oscillations increases smoothly with γ, and for $\ell > 2x_1$ the change in amplitude is brutal. There is an intermediary region with a smooth increase of the amplitude followed by a brutal jump. This could indicate a change in the nature of the Hopf bifurcation, from super-critical to sub-critical. To verify this idea, we use harmonic balance, and continuation in function of γ for 3 particular values of ℓ : $\ell = x_1 = 0.4$, $\ell = 0.6$ m and $\ell = 1$ m.

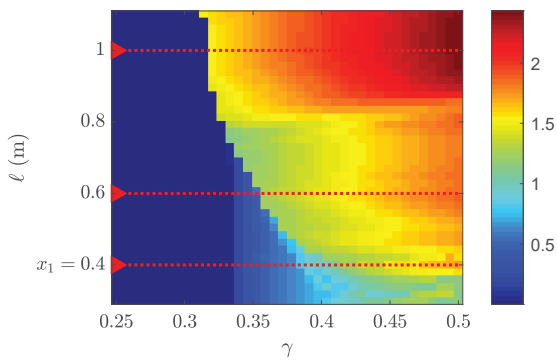

Figure 1. Cartography of the final peak-to-peak amplitude of the internal pressure signal p. Red dotted lines: particular values of ℓ where the bifurcation is studied by continuation (figures 3 and 2).

2.3 Continuation with harmonic balance method

A Taylor-series based continuation method, called asymptotic numerical method (ANM), is used. It is based on the numerical continuation of a system of equations

$$R(X,\lambda) = 0 \quad \text{where } X, R(X,\lambda) \in \mathbb{R}^n \text{ and } \lambda \in \mathbb{R}. \tag{10}$$

The parameter λ is the continuation parameter and the vector of unknowns is X. The solution-set of equations (10) can be represented as a collection of (one-dimensional) curves in the total space (X,λ). The ANM has the specificity to require a specific treatment of the equations to obtain an equivalent system of equation with at most quadratic nonlinearities [1]. This allows to compute an approximation of the solution branches $(X(a),\lambda(a))$ on the form of Taylor series

$$X(a) = X_0 + X_1 a + X_2 a^2 + \dots \quad \text{and} \quad \lambda(a) = \lambda_0 + \lambda_1 a + \lambda_2 a^2 + \dots \tag{11}$$

where a is a parametrization of the solution-branch which verifies here $a = X_1(X - X_0) + \lambda_1(\lambda - \lambda_0)$. In the present work, the equation of the model is discretized in the frequency domain using harmonic balance method (HBM), that is a truncated Fourier expansion of the unknown

$$p(t) = p_0 + \sum_{k=1}^{H} p_{c,k}\cos(k\omega t) + p_{s,k}\sin(k\omega t) \tag{12}$$

This ansatz is put in the quadratic recast of the model equation in order to obtain a system of the form (10) where X is a vector containing the all Fourier coefficients and the unknown angular frequency ω. The parameter λ is γ or ℓ in the following study. The truncated Fourier expansion (at order $H = 50$ in the applications) of p and the angular frequency ω are then known along the solution branch. The generalization of the coupling between HBM and ANM in the case of systems with time-delay has been done in [2].

3 Results

The results in this section are presented on the form of several figures. On these figures, the amplitude and the angular frequency of the acoustic pressure p computed with the continuation method are represented in blue solid lines. The truncation order for the Fourier series is $H = 50$. The red crosses are the point-wise periodic solutions given by the sound synthesis algorithm. The red circles are quasi-periodic steady state solutions. The comparison is made for values of the parameters along the red dotted lines of the figure 1. The agreement of both methods is very good on most of the results. A surprising feature of the system at ℓ fixed is that even if the Hopf bifurcation seems super-critical in all cases, the larger ℓ is, the sooner the Hopf bifurcation is followed by a first fold bifurcation (or limit point), before the second one where the periodic solution retrieves its stability. For ℓ sufficiently small (see right figure 3) the two fold bifurcations are getting closer and with less effect on the dynamics of the system. However, in this case the periodic solution seems to become unstable quickly to the benefit of a quasi-periodic solution obtained with sound synthesis. In terms of frequency, a pitch-flattening effect is observed for high γ values, but before the first fold bifurcation, the pitch rises when γ augments.

4 CONCLUSIONS

For this simple saxophone model, the Hopf bifurcation is never found to be sub-critical, but hysteretic behavior can appear due to the presence of two fold bifurcations, which are increasingly far apart as the length of the resonator augments. Further work may include a study of the dependence of these phenomena on the volume of the mouthpiece or the reed opening control parameter ζ, as well as a study on the stability of the equilibrium and the oscillating solution in the region where they coexist.

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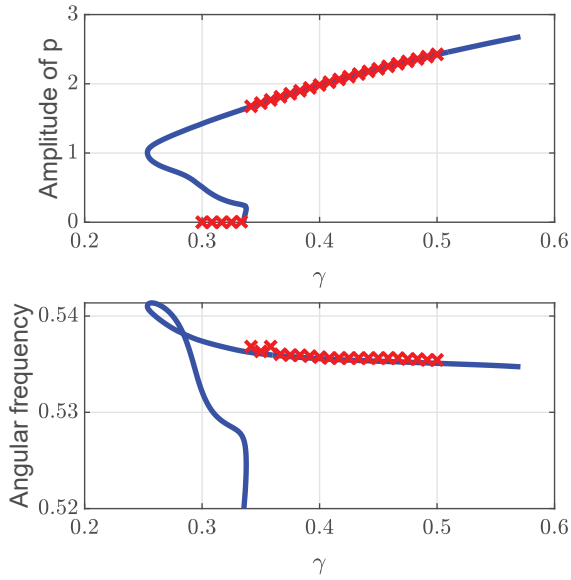

Figure 2. Bifurcation diagram representing the final peak-to-peak amplitude and the frequency of the internal pressure signal p, for $\ell = 1$. Blue line: continuation and harmonic balance. Red crosses: sound synthesis.

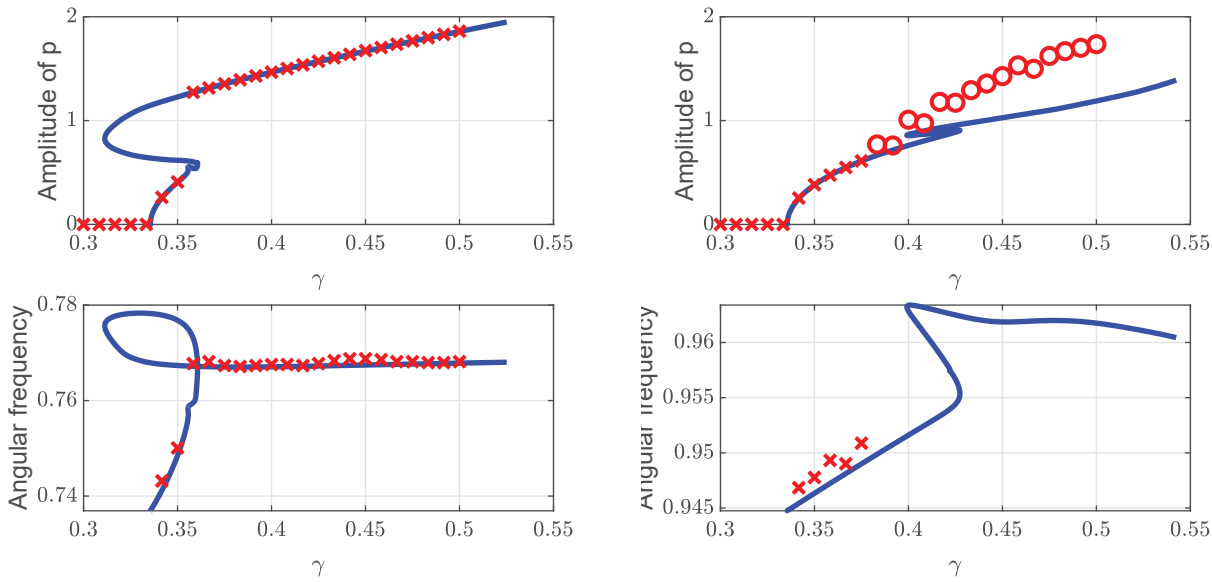

Figure 3. Bifurcation diagram representing the final peak-to-peak amplitude and the frequency of the internal pressure signal p. Blue line: continuation and harmonic balance. Red crosses: sound synthesis. Left : $\ell = 0.6$. Right : $\ell = 0.4$.

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