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The Analysis of Parabolic Equation Model Solutions using Split-Step/Finite Difference Method

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ABSTRACT

Parabolic equation models solved using the split-step Fourier (SSF) algorithm is commonly studied for the modeling of wave propagation in the underwater environment. The smoothing function, in the solution of realistic problems, is introduced in the SSF algorithm to treat the density change between the water column and the sediment layer. The algorithm with the smoothing approach accumulates the phase error with the range. As an alternative method, by Yevick and Thomson, the hybrid split-step/finite difference method was developed to accomplish the density discontinuity issue numerically, and it significantly improved the phase error. The additional operator containing density dependent terms are introduced to implement the hybrid approach in this method. The results of the SSF parabolic equation model with the smoothing approach and with the hybrid approach are compared and discussed to see the effect of numerical parameters on the model implementation in this research.

Keywords: Parabolic equation, Split-Step Fourier algorithm, Smoothing function, Density discontinuity, Split-step/finite difference method.

1. INTRODUCTION

Parabolic equation (PE) approximation to the modeling of radio wave propagation in the atmosphere was introduced by Leontovich and Fock (1). It applies to various fields related to wave propagation such as optics, laser-beam propagation, microwave propagation and seismic propagation (2). It was used for the modelling of acoustic wave propagation in underwater acoustic, and PE (3,4) was solved by the SSF algorithm as a numerical technique. Alternatively, the techniques, such as finite-difference (FD) and finite-element (FE), are also applied to PE to model the ocean acoustic propagation (5,6).

PE method, introduced by Tappert, is known small-angle PE (SPE) because the model gives the increase of error with increasing of propagation angle. For more accurate solutions at higher angles of propagation, the wide-angle parabolic equation (WAPE) method was introduced. The solutions of WAPE with SSF model gives more stable and efficient solutions (7,8).

The general form of PE, the first order differential equation, is obtained by introducing the acoustic field function in Helmholtz equation. It was a simplified differential form that expressed in terms of the operator called the square root operator. The reduced wave equation contains the index of refraction which has the density and its derivative terms. In numerical solutions, the terms related to the density are approached to describe the density discontinuity caused by the vertical density variations at interface between two half space. Various numerical models treat density discontinuity in differential form of wave equation by different approach. The boundary discontinuity leads to phase error with range in SSF numerical solutions and the appropriate approaches can be treated to decrease the numerical errors (9). In PE/SSF methods, the smoothing functions such as the hyperbolic functions are implemented to treat the boundary discontinuity density, but solutions accumulates the phase error with the range, so by Yevick and Thomson (10), the hybrid split-step/finite difference method was developed to treat the density discontinuity issue, and it significantly improved the phase error.

In this paper, we will examine the smoothing approach and split-step Fourier / finite-difference (SSF/FD) approach. The results of PE/SSF model with smoothing functions and with hybrid approach will be compared to benchmark solutions for shallow water environments.

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2. THEORY

2.1 The Parabolic Equation (PE) Approximation for Acoustic Wave Propagation

For the acoustic wave propagation, the complex pressure is defined as a solution with time harmonic at frequency *f* in following equation.

$$P(r,\varphi,z,\omega t)=p(r,\varphi,z)e^{-i\omega t} \tag{1}$$

This equation is substituted in the Helmholtz equation, and it leads to another equation for underwater acoustic propagation that is given by free-field Helmholtz equation in the following form with 3-D cylindrical coordinates,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 p}{\partial \varphi^2}+\frac{\partial^2 p}{\partial z^2}+k_0^2 n^2(r,\varphi,z)p=0, \tag{2}$$

where $n(r,\varphi,z)=c_0/c(r,\varphi,z)$ is the acoustic index of refraction, $k_0=2\pi f/c_0$ is reference wave number with frequency *f*, c_0 is the reference sound speed.

In far field propagation, the media is assumed as a cylindrical waveguide and the pressure field is proportional to reversed by $\sqrt{r}$ because of energy conserving. To obtain more general PE form, the field function ψ is defined by the acoustic pressure field along a single radial as,

$$\psi(r,z)=\sqrt{k_0 r}\,p(r,z)e^{-ik_0 r} \tag{3}$$

The PE is obtained by the substituting the field function into the Helmholtz equation in cylindrical coordinates,

$$\frac{\partial \psi}{\partial r}=ik_0\left(Q-1\right)\psi \tag{4}$$

where Q is generally called as the "square-root operator,"

$$Q=\sqrt{n^2+\frac{1}{k_0^2}\rho\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial}{\partial z}\right)} \tag{5}$$

where $n(r,z)=c_0/c(r,z)$ is the acoustic index of refraction and c_0 is typically assumed as 1500 m/s. In the studies of modeling of underwater acoustic wave propagation based on PE, the first order differential equation given above is numerically solved using various methods.

As an alternative approach taking account of the influence of density in the Helmholtz equation, the field function is defined in the following form (8),

$$\tilde{\psi}=\sqrt{\rho}\psi \tag{6}$$

which is substituted in the Helmholtz equation, and a PE form is also satisfied by this approach,

$$\frac{\partial \tilde{\psi}}{\partial r}=ik_0\left(\tilde{Q}-1\right)\tilde{\psi} \tag{7}$$

The second order differential equation, the Helmholtz equation, is reduced to the first order differential equation that can be solved one-way marching algorithm which is given by

$$\tilde{\psi}(r+\Delta r,z)=\exp\left[ik_0\Delta r(\tilde{Q}-1)\right]\tilde{\psi}(\mathrm{r},z) \tag{8}$$

where the operator $\tilde{Q}$ contains an effective index of refraction,

$$\tilde{n}=\sqrt{n^2+\frac{1}{2k_0^2}\left[\frac{1}{\rho}\frac{\partial^2\rho}{\partial z^2}-\frac{3}{2}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial z}\right)^2\right]} \tag{9}$$

In this approach, the numeric solution treats the density contrast into the effective index of refraction.

2.2 Split-Step Fourier (SSF) Method

The SSF method based on a marching algorithm which can be represented by the following form,

$$\psi(r+\Delta r,z)=\Phi(r)\psi(r,z) \tag{10}$$

where $\Phi(r)$ is propagator function, PE can be written in terms of Hamiltonian-like operator, H_{op}.

$$\frac{\partial\psi}{\partial r}=-ik_0H_{op}\psi \tag{11}$$

for the numerical solution, the Dyson evolution operator (11) is introduced by

$$\tilde{\psi}(r+\Delta r,z)=\Im\exp\left[-ik_0\int_r^{r+\Delta r}dr'H_{op}(r')\right]\psi(r) \tag{12}$$

where the expansion is analyzed by lower-order corrections and higher-order corrections is neglected. The propagator operator advanced the solution with small range step is

$$\Phi(r)\approx\exp\left[-ik_0\Delta r\bar{H}_{op}(r)\right] \tag{13}$$

where range-averaged operator, $\bar{H}_{op}(r)$,

$$\bar{H}_{op}(r)=\frac{1}{\Delta r}\int_r^{r+\Delta r}dr'H_{op}(r') \tag{14}$$

and H_{op} consists of scalar and differential operators, and these operators could be implemented by multiplication, so the basis of SSF algorithm is based on separated terms of H_{op} and approximations to the square-root operator. In the model (8), the azimuthal coupling term is assumed small, and that is incorporated by binomial expansion. For the terms within the square root, the WAPE approximation of Thomson and Chapman (7) is utilized.

$$\tilde{Q}=\sqrt{\tilde{n}^2+\frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}}\to\tilde{Q}_{WAPE}=\sqrt{1+\mu}+\sqrt{1+\tilde{\varepsilon}}-1 \tag{15}$$

$$\tilde{\varepsilon}=\left(\tilde{n}^2-1\right),\ \mu=\frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} \tag{16}$$

The complete approximation for H_{op} is

$$H_{op}\approx T_{op}+U_{op}+V_{op}, \tag{17}$$

where T_{op}, U_{op} and V_{op},

$$T_{op}=1-\left[1+\frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right],\ \ U_{op}=-(n-1),\ V_{op}=-\frac{1}{2k_0^2r^2}\frac{\partial^2}{\partial\varphi^2} \tag{18}$$

For range-independent environment, operator $\bar{H}_{op}(r)$ can be chosen as $\bar{H}_{op}(r)\approx H_{op}(r)$ which can be simply expressed in terms of operators T_{op} and U_{op} (due to azimuthal symmetry, $V_{op}\to 0$)

$$\Phi(r)=e^{-ik_0\Delta rT_{op}\left(\frac{\partial^2}{\partial z^2}\right)}e^{-ik_0\Delta rU_{op}(r)}\ \text{ or }\ \Phi(r)=e^{-ik_0\frac{\Delta r}{2}U_{op}(r+\Delta r)}e^{-ik_0\Delta rT_{op}\left(\frac{\partial^2}{\partial z^2}\right)}e^{-ik_0\frac{\Delta r}{2}U_{op}(r)} \tag{19}$$

In SSF algorithm, U_{op} is calculated by a multiplication operation in z-space, but the calculation of operator T_{op} is not a simple operation that the field is transformed to vertical wave number space k_z. SSF method is essentially based on the separation of influence of the operators U_{op} and T_{op}. In the fundamental scheme of this algorithm, the field $\psi(r,z)$ is firstly defined over depth at some range (r), then the operator $e^{-ik_0\Delta rU_{op}(r)}$ is applied on the initial field, and the result field is transformed by Fourier transformation. The transformed field is multiplied by k-space operator, $e^{-ik_0\Delta r\hat{T}_{op}\left(k_z^2\right)}$. The k-space field is transformed back to z-space, and the solution field, $\psi(r+\Delta r, z)$ is obtained at the end of process.

$$\psi(r+\Delta r, z) = IFFT\left\{e^{-ik_0\Delta r\hat{T}_{op}(k_z)} \text{ x } FFT\left[e^{-ik_0\Delta r U_{op}(r,z)} \text{ x } \psi(r,z)\right]\right\} \tag{20}$$

2.3 Boundary Conditions and Smoothing Approach in SSF Method

In PE/SSF model, the surface boundary is assumed as a perfect reflector to provide the pressure release boundary condition, $\psi(z=0)=0$, and the image ocean method is used to realize the boundary condition. In image ocean method, the real ocean is in the positive z-direction, and the image ocean lays over the real ocean in the negative z-direction.

$$\psi(-z) = -\psi(z) \tag{21}$$

Eq. (21) shows that the acoustic field of image ocean is equal to opposite sign of the acoustic field of the real ocean. In bottom boundary, the discontinuity occurs between two different media due to the contrasting sound speed and density. These discontinuities create the numerical noise in the SSF algorithm, so a function is needed to treat the discontinuity. The operator U_{op} depends on the effective index of refraction term which includes the sound speed discontinuity and the density discontinuity in bottom interface. The discontinuities may describe as follows

$$c(z) = c_{water}(z) + \left(c_{bottom}(z) - c_{water}(z)\right)H\left(z - z_{bottom}\right) \tag{22}$$

$$\rho(z) = \rho_{water} + \left(\rho_{bottom} - \rho_{water}\right)H\left(z - z_{bottom}\right) \tag{23}$$

The terms in the effective index of refraction Eq. (9) depend on the density change, $\rho(z)$, it is also needed to a generalized function to represent the differential terms $\frac{\partial \rho}{\partial z}$ and $\frac{\partial^2 \rho}{\partial z^2}$ in numerical calculation. This mixing function is introduced analytically (8) by defining the density change at the water-bottom interface according to the density profile Eq. (23).

2.4 The Hybrid Split-Step Fourier / Finite Difference Approach

For density discontinuity in bottom layer interface, the solutions of SSF algorithm based on density smoothing approach have phase error issue to be produced with range. Instead of smoothing approach, Yevick and Thomson (10) were developed a hybrid technique which is the relying on the solving of density dependent terms by finite difference calculations. The implementation of this approach can also be adapted to existing SSF codes. They introduced another operator to split the operators as density dependent and density independent that the density dependent operator is effectively applied by using finite difference technique with Pade approximation. In this approach, as the density dependent operator, γ is defined and the modified square root operator is considered to be of the form

$$Q = \sqrt{1+\mu+\varepsilon+\gamma} \tag{24}$$

where $\gamma = \frac{1}{k_0^2 \rho}\frac{\partial \rho}{\partial z}\frac{\partial}{\partial z}$ and now density discontinuity is treated by finite difference technique applied on the operator γ, but the sound speed discontinuity in the layer will be still used the same treatment without any modification as before. WAPE approximation is applied on new modified square-root-operator that gives

$$Q_{WAPE} = \sqrt{1+\gamma} + \sqrt{1+\mu} + \sqrt{1+\varepsilon} - 2 \tag{25}$$

Thus, for the solution of one-way marching algorithm given in Eq. (8), an additional operation is required due to a new operator γ, which is

$$\psi(r+\Delta r, z) = \exp\left[ik_0\Delta r(\sqrt{1+\gamma}-1)\right]\tilde{\psi}(r+\Delta r, z) \tag{26}$$

The additional operation is realized by a first-order Padé expansion. In the final step, finite-difference approximation is applied to the expression defined by Eq. (20).

$$\left[1+\frac{1}{4}(1-i\delta)\gamma\right]\Psi(\mathrm{r}+\Delta\mathrm{r},\mathrm{z})=\left[1+\frac{1}{4}(1+i\delta)\gamma\right]\tilde{\Psi}(\mathrm{r}+\Delta\mathrm{r},\mathrm{z}). \tag{27}$$

If γ operator is directly implemented, it causes problematic calculation process. Yevick and Thomson introduced an alternative operator to accomplish this issue and to solve the finite difference equations effectively.

3. RESULTS

For numerical calculations, the shallow water environment is assumed to have flat surface and bottom boundary at a depth of 300 m from the water surface. The calculation environment consist of water column with sound speed of c_w=1500 m/sec, density of ρ_w=1.0 g/cm^3 and fluid-like sediment with sound speed of c_s=1700 m/s, density of ρ_s=1.5 g/cm^3. The benchmark solution is required the results obtained from normal mode model (12) for our solutions which have the environment with flat, pressure-release surfaces.

Transmission loss is modeled for signals transmitted from a point source at zero range with a single frequency of 100 Hz at a depth of 180 m. Transmission loss traces are computed for a receiver located at a distance of 20 km from source and at a depth of 100 m. The discretization values of environment for numeric calculation are described by the range step size (*dr*) and depth mesh size (*dz*) and also defined fast Fourier transform (FFT) size (*NZ*). The scaling factor (*rz*) is a multiplier between the *dr* and *dz*. The depth mesh size was defined to be associated with acoustic wave length which is the range of $\lambda/6$–$\lambda/30$. Subsequently, the range step size was defined as some integer multiplier of *dz*.

Figure 1,2 and 3 display the solutions of basic SSF algorithm that uses the smoothing approach. Different values of reference sound speed c_0, scaling factor *rz* and depth mesh size *dz* were used with the goal of producing good match with the benchmark solution. The model results match relatively well with the benchmark solution for c_0=1490, but the solutions in still accumulate phase error with the range depending on the different values of c_0, *rz* and *dz*.

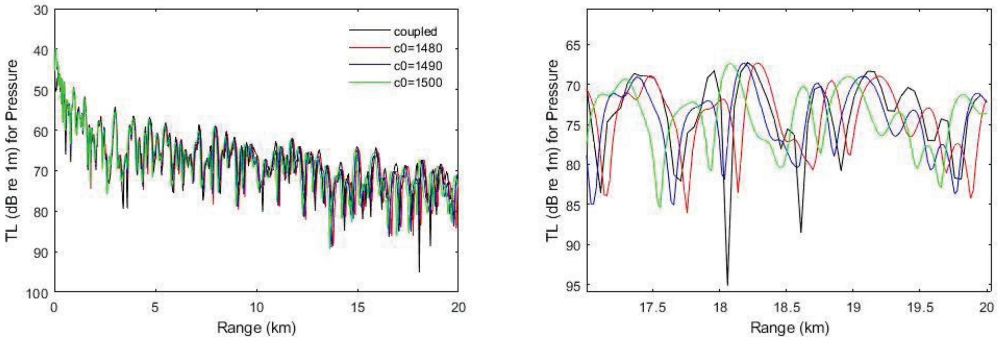

Figure 1 – Smoothing approach with *rz*=5, *dz*=1,025 m for different values of sound speed c_0

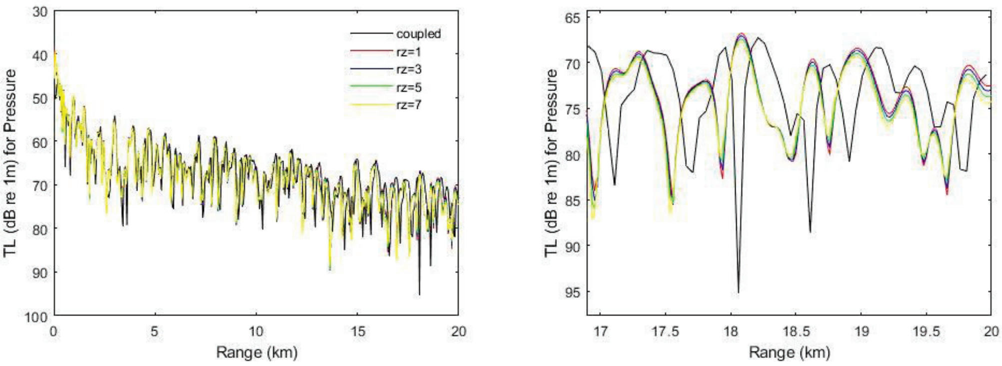

Figure 2 – Smoothing approach with c_0=1500 m/s, *dz*=1,025 m for different values of scaling factor *rz*

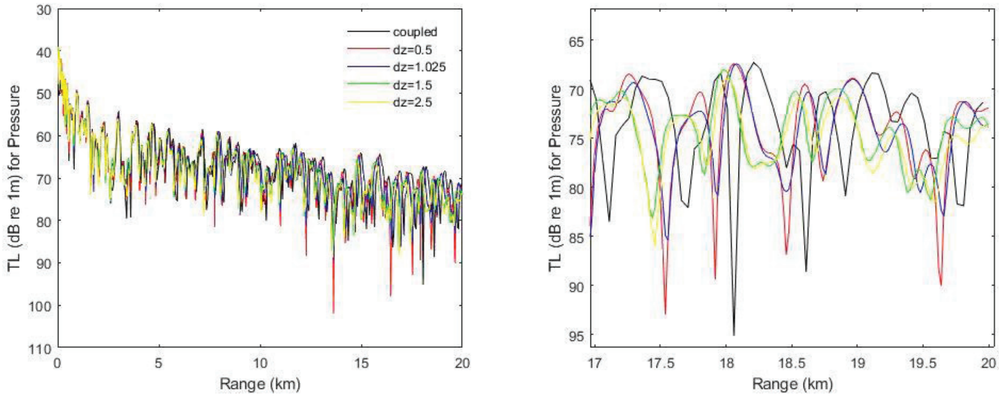

Figure 3 – Smoothing approach with c_0=1500 m/s, *rz*=5 for different values of depth mesh size *dz*

In hybrid SSF/FD algorithm, γ operator is implemented before the operator term containing ε and after operator containing term μ and the result is given by red trace of transmission loss in Figure 4. γ is also applied after or before from both ε and μ to test the effect of operator order and the result is given by blue trace of transmission loss in in Figure 4.

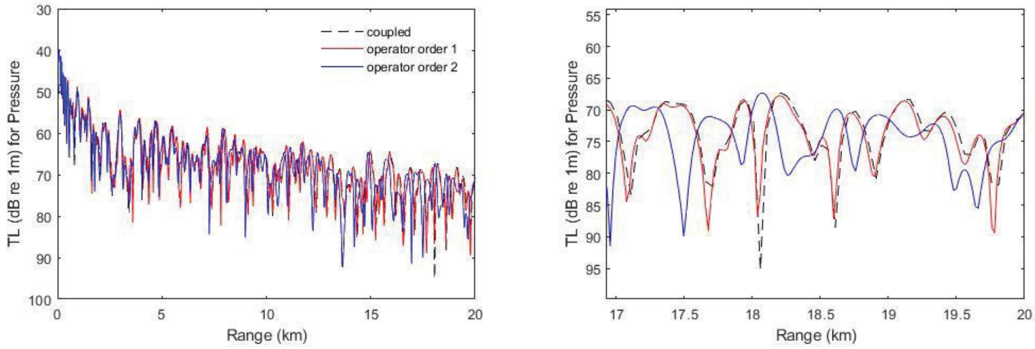

Figure 4 – Hybrid SSF/FD approach related to operator order

The solutions treated the hybrid SSF/FD approach in the presence of the bottom interface density change is displayed in Figures 5, 6 and 7. The model results match well with the benchmark solution for c_0=1500 in Figure 5. Figure 6 shows that the solutions don't importantly change with scaling factor *rz*.

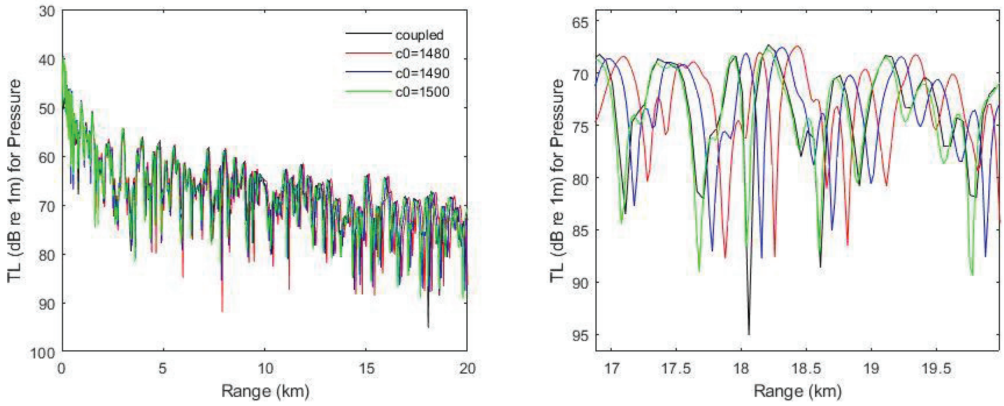

Figure 5 – Hybrid approach with *rz*=5, *dz*=1,025 m for different values of sound speed c_0

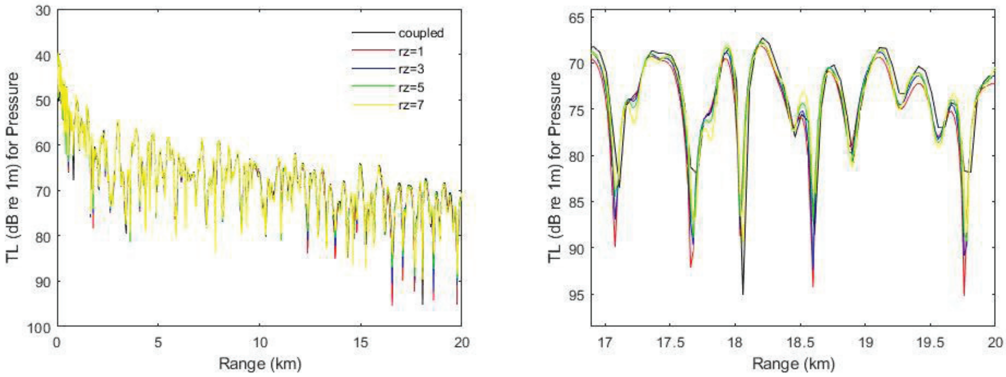

Figure 6 – Hybrid approach with c_0=1500 m/s, *dz*=1,025 m for different values of scaling factor *rz*

Figure 7 shows that the transmission loss trace at receiver depth of 100m for different depth mesh values (dz = 0.5m, 1.025m 1.5m and 2.5m) to be clear the effect of depth mesh size on the solution and the best solution is at depth mesh size value of dz=1.025m.

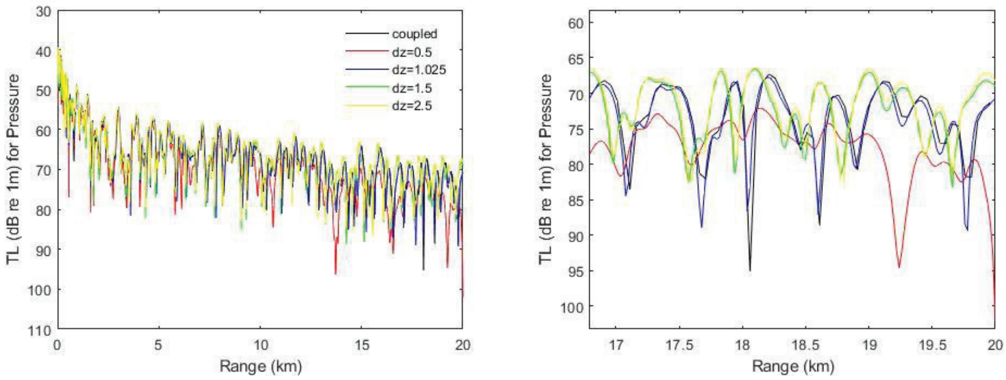

Figure 7 – Hybrid approach with c_0=1500 m/s, *rz*=5 for different values of depth mesh size *dz*

It can be observed from Figure 8 that the results from SSF/FD approach gives well agreement with the benchmark solution, but SSF approach shows some phase and amplitude errors at certain ranges. SSF/FD approach to boundary density discontinuity treats phase error for a depth mesh size of dz = 1.025 m.

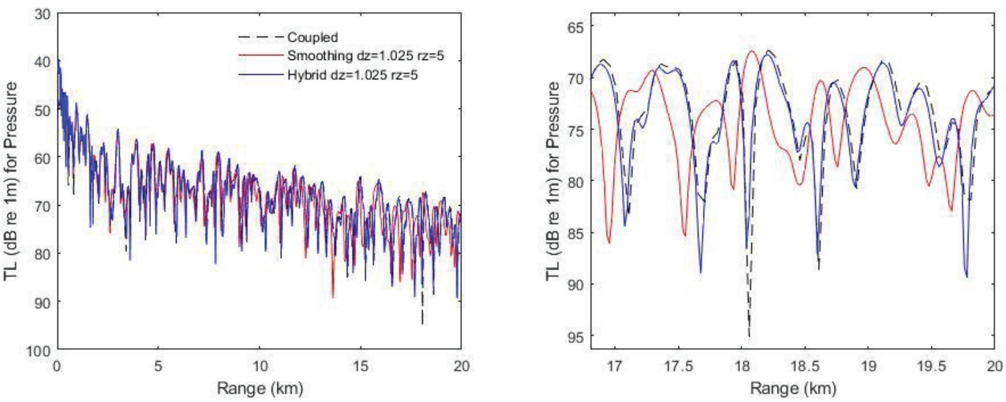

Figure 8 – Density smoothing approach and Hybrid approach compared with benchmark solution

4. CONCLUSIONS

The figures have two panels that the second panel shows clearly phase errors. The panel in left shows full range, and the panel in right shows expanded view of final 3 km. It was shown that SSF algorithm gives well solutions for a few kilometers, but then it accumulates phase error with the range. Model solutions are sensitive to reference sound speed $c_{0.}$ The solutions converge to benchmark solution for c_0=1490m/s, but the solutions significantly give the phase error.

Alternatively, the bottom density discontinuity was treated hybrid approach that was introduced by Yevick-Thomson. Thus, standard SSF algorithm was updated with a hybrid SSF/FD algorithm that improves the solutions. We take into account the operator order for operator applying in Eq. (20). SSF/FD algorithm was compared to the benchmark solution and it was shown that produces high quality solutions for depth mesh size value of dz=1.025 m, sound speed value of c_0=1500 m/s and first choice of operator order. It is also shown that both the solutions of density smoothing approach and the solutions of hybrid SSF/FD approach are not affected by the choice of scaling factor.

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