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Vibration modelling of an elastic body of arbitrary shape subjected to mixed excitation

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ABSTRACT

Using the known analytical Green's function for the displacements and stresses of a force-excited infinite, elastic, homogeneous solid the response of a finite body of arbitrary shape can be reconstructed. The response is obtained by applying to the infinite solid a distributed force excitation in the exterior and interior of a virtual closed surface which coincides with the surface of the targeted finite body. The role of such an excitation is to re-create across this surface the boundary conditions of the finite body. Such an approach permits one to obtain the response of the finite body under rather general excitation conditions. The latter may consist of a superposition of kinematic movements applied over one part of the body surface and the distributed force excitation within the body. The physical side of the boundary forming approach is discussed first, the corresponding mathematical procedure is then outlined and examples are provided to demonstrate its validity.

Keywords: Vibration, Modelling

1. INTRODUCTION

An analytical or a semi-analytical solution can be useful in cases where the physical insight into the nature of vibration is needed. Straightforward analytical solutions to vibration response exist only for objects of very simple shape and boundary conditions, such as a beam, plate or shell having simple supported terminations. The computation of vibration response of a mechanical object of complex shape is usually done by a numerical approach, notably by the Finite Element Method.

A technique termed the Boundary Forming Approach allows for a semi-analytical vibration modelling of beams and plates of arbitrary shape, (1). The model requires the Green's function of the corresponding body of infinite size. The technique is a satellite version of an approach called the DPS method used so far primarily for the modelling of ultrasonic and electromagnetic fields (2). The boundary forming technique was recently applied to vibration modelling of a 3D homogeneous beam subjected to force excitation, (3).

The frequency-domain Green's function relative to displacement vector in an arbitrary point of an isotropic and homogeneous infinite medium is known, (4). This enables construction of the solution to vibration response of a body of arbitrary shape and quite complex boundary conditions subjected to both dynamic and kinematic type of excitation.

Since the Green's function is defined in frequency domain the sinusoidal temporal variation is assumed, with the amplitudes of excitation and response quantities taken as complex in the usual way. The entire modelling will be thus restricted to operations with complex amplitudes exclusively; the common time-variation factor will be omitted.

2. APPROACH

2.1 The principle

The original vibrating body is supposed to be excited by an excitation which may consist of both a dynamic component, involving distributed forces and moments, and a kinematic component, involving distributed displacements and rotations. The kinematic component is supposed to act across either the entire surface of the body or only one part of it. The part of body surface not subjected to kinematic excitation may have one area immovable while the remaining area then remains free.

The vibration will be entirely modelled by means of an infinite medium within which a virtual boundary surface is defined, corresponding to the physical surface of the original body. A double excitation is applied to the infinite medium: a primary one and a secondary one. The primary excitation is the same as the dynamic component of the original excitation and occupies the same position with respect to the virtual boundary as does the original excitation with respect to the physical boundary. The secondary excitation has the role of reconstructing, in parallel with the primary excitation, the original boundary conditions and kinematic excitation across the virtual surface. Since the vibration field within the boundary surface contains only the primary excitation, the secondary excitation has to be placed outside the volume delimited by the boundary surface. In such a case the part of the infinite medium within the boundary, artificially created by the primary and secondary excitation, will behave exactly as if this part was independent of the rest and driven by the original excitation, Fig. 1.

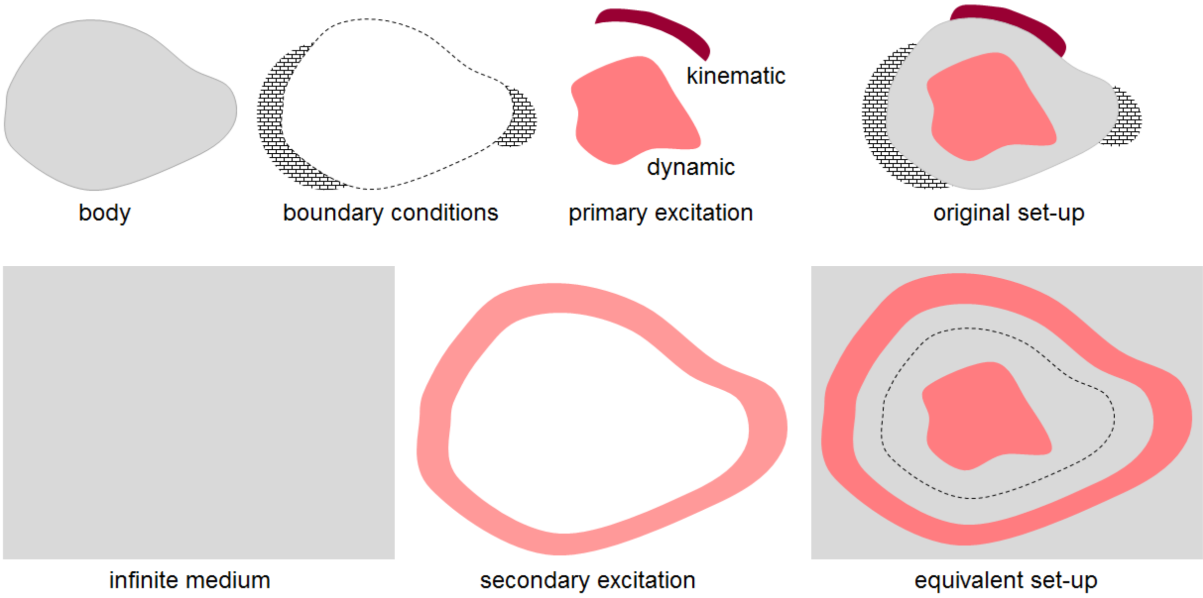

Figure 1 – The original vibration body (top) and its model (bottom).

While the excitation is assumed to be distributed continuously across the excitation domain, the computation, using the discrete-point Green's function, will require the excitation be approximated by a discrete one, distributed over a large number of (densely-spaced) excitation points. The Green's function used in the model represents the displacement response to a unit sinusoidal point force. A single point force produces two types of elastic waves: a dilatational one and a shear one. The elastic waves created by the primary sources will superpose with the waves created by the secondary sources. In steady state motion the interference between the primary and secondary waves may create in a number of points particular conditions, such as a given displacement or a given stress. Providing the positions of secondary sources are conveniently selected and their strengths correctly identified the interference between the primary and secondary waves will create the required boundary conditions exactly at points located across the virtual boundary. If these points, termed the *control points*, are sufficiently close to each other and not separated by singularities the boundary conditions in between control points will closely match the required conditions too.

2.2 Field variables

The vibrating body is supposed to be made of an elastic, homogeneous, isotropic solid of mass density ρ, Young's modulus E and Poisson's factor ν. Harmonic excitation at a frequency ω is assumed; it becomes thus sufficient to operate only with the complex amplitudes of various field variables. The orthogonal components of displacement response $\mathbf{U} = [U_1,U_2,U_3]^t$ at an arbitrary point $\mathbf{d} = [d_1,d_2,d_3]^t$ are related to the orthogonal components of the excitation force $\mathbf{F} = [F_1,F_2,F_3]^t$ at a point $\mathbf{e} = [e_1,e_2,e_3]^t$ by the 3×3 Green's function of the solid $\mathbf{G}$, (4):

$$\mathbf{U} = \mathbf{G}(\mathbf{d},\mathbf{e})\mathbf{F}, \qquad G_{mn} = \frac{\xi^p_{mn} + \xi^s_{mn}}{4\pi\rho\omega^2}, \quad m,n = 1,2,3 \tag{1}$$

Using the symbols c_p for the speed of dilatational waves, c_s for the speed of shear waves and δ for the Kronecker delta function the ξ terms in Eq. (2) read:

$$\xi_{mn}^{p} = \frac{e^{jk_p r}}{r}\left[k_p^2\gamma_{mn} + (3\gamma_{mn} - \delta_{mn})\left(\frac{jk_p}{r} - \frac{1}{r^2}\right)\right]$$

$$, \quad \xi_{mn}^{s} = \frac{e^{jk_s r}}{r}\left[k_s^2(\delta_{mn} - \gamma_{mn}) - (3\gamma_{mn} - \delta_{mn})\left(\frac{jk_s}{r} - \frac{1}{r^2}\right)\right] \tag{3}$$

$$r = |\mathbf{d} - \mathbf{e}|, \quad \gamma_{mn} = \frac{(d_m - e_m)(d_n - e_n)}{r^2}, \quad k_p = \frac{\omega}{c_p}, \quad k_s = \frac{\omega}{c_s}$$

The stress tensor in the point **d** can be expressed in terms of spatial displacement derivatives by the differential relationship between stresses and displacements. By denoting the strain components with $u_{mn} = \partial u_m/\partial x_n$ and assuming summation over repeating indices the components of stress tensor read:

$$\sigma_{mn} = \frac{E}{1+\nu}\left(u_{mn} + \frac{\nu}{1-2\nu}u_{ii}\delta_{mn}\right) \tag{4}$$

2.3 The vibration model

The boundary forming approach applied to vibration of a 3D body requires that the primary and secondary excitations are represented by discrete forces. The state across the boundary, i.e. at the virtual boundary surface within the infinite solid, is observed at a number of control points on this surface. The number and the positions of both the secondary sources and control points on the boundary surface need to be selected in some way. The positions of these points cannot be found from any definite physical consideration; the selection has to be done by intuition or by a convenient search procedure. While too few points may not well represent the otherwise continuous boundary conditions, too many points may create difficulties in computation and unacceptably increase the execution time. Once the positions of points have been selected the computation of secondary sources becomes a fairly straightforward mathematical task based on matrix operations. Assuming C control positions and S secondary source positions the following procedure can be applied:

- Define the boundary conditions via either displacements, stresses or both at the control points. E.g. if the boundary variables are surface stresses, three stress components have to be taken into account per each control point: the out-of-plane normal stress and two in-plane shear stresses. If instead the variables are the displacements, there will be three displacement components per boundary point. Thus in both cases the number of boundary conditions per point is $B = 3$.
- Specify the targeted values of boundary variables. The targeted values can be arranged into a vector **T** the length of which is thus $K=B\cdot C$ – the number of boundary variables. E.g. across the area of the body surface which is free the three components of surface stress will have to vanish across the contour surface. If some parts of the body surface are driven by kinematic excitation the three displacement components per each boundary point have to be specified.
- Compute the transfer functions between the boundary variables and all the secondary sources. Since there are S secondary sources and three force components per source a transfer matrix **H** will be thus obtained of size $K\times3S$.
- Compute the values of the boundary variables at the control points due to the dynamic primary sources (the kinematic primary excitation is accounted for via the boundary vector **T**). A vector of primary boundary variables $\mathbf{V}_0$ can be thus formed, the length of which is, just like that of **T**, equal to K.
- The strengths of secondary sources arranged in a vector **Q** of size $3S$ then reads:

$$\mathbf{Q} = \mathbf{H}^{-1}(\mathbf{T} - \mathbf{V}_0) \tag{5}$$

The vector of secondary source strengths **Q** contains three orthogonal complex force amplitudes per point: its length is thus $3S$. Once the vector **Q** has been found, the response of the vibrating object, limited by the virtual boundary surface, can be obtained by superposing the fields created by the primary and secondary excitations.

The technique described needs discretisation of both primary and secondary excitation in order to enable numerical implementation of Eq. (5). The results are thus not exact but approximate. However, in contrast to traditional numerical techniques based on discretisation, such as FEM, here only the excitation is discretised while the response field remains continuous. This may be of benefit where fine detail of the field is needed.

The technique requires matrix inversion as seen by Eq. 5.. Theoretically the number of boundary variables K and the number of secondary force components $3S$ should be equal in order to allow for matrix inversion in Eq. 5. Such a condition will lead to poorly conditioned matrix **H**. This may result in values of secondary forces which, while producing at the control points the boundary conditions close to the targeted ones, may produce large deviation from boundary conditions in between these points. In order to improve the result the number of boundary variables should be made larger than that of secondary force components, $K > 3C$. In such a case the Eq. 5 will represent an overdetermined case: the matrix **H** will not be square and the inversion should be replaced by pseudo-inversion. The latter can be done using one of techniques of solving overdetermined problems, e.g. the Moore-Penrose pseudo-inverse or the minimisation of a norm of $\mathbf{HQ}+\mathbf{V}_0-\mathbf{T}$.

3. EXAMPLES

3.1 The vibrating body

The boundary forming technique was examined in (3) using free boundary conditions. The present paper extends it to a more intricate case of mixed conditions. Like in (3) the vibrating object is a beam of moderate thickness. The beam will be driven by an externally imposed kinematic excitation applied to one end. The Euler-Bernoulli beam model will be used to provide an analogous 1D “reference” case for comparing it with that obtained by the boundary forming approach.

The beam dimensions are: length 0.6m, width 12cm, thickness 2cm. It is made of steel: $E = 2\times10^{11}$ Pa, $\rho = 7800$ kg/m3, $\nu = 0.3$. The loss factor of 0.01 is assumed; it is input in the model by assigning the corresponding complex value to the Young’s modulus of the medium.

In the present case the contour surface, identical to the surface of the beam, is a simple parallelepiped. The spacing between control points was selected at 8.5mm, which resulted in 2422 control points. The secondary sources were placed across two surfaces of shape similar to that of the contour surface but displaced from it outwards by 100mm and 250mm with the spacing between source points of 40mm and 80mm respectively.

The selected positions and spacing between the secondary sources produced the total number of 1130 source points. The layout of different points is shown on Fig. 2.

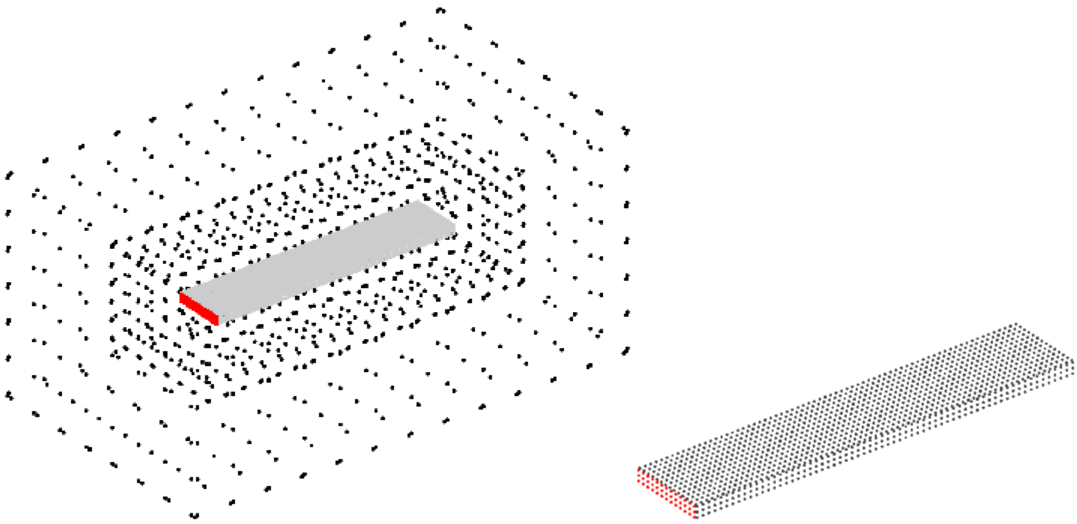

Figure 2 – Secondary source points (left) and control points (right) of the beam model. Excited face in red.

The selected parameters, i.e. $C = 2422$ and $S = 1130$, result in a matrix H of size $K\times3S = 7266\times3390$. As a rule, the higher the frequency, i.e. the smaller the wavelength of the two types of waves in the solid, the smaller the needed spacing between the points. The selected parameters were obtained by keeping the total number of points reasonably modest in order to allow for a PC-level computation.

3.2 Response to kinematic excitation

To start with consider the beam driven at the left end by a vertical displacement of 5mm amplitude at 500 Hz. The analytical solution in closed form can be obtained for this case using the simple Euler-Bernoulli beam model. Fig. 3 shows the RMS value of vibration amplitude computed by the analytical model and the 3D boundary forming approach. The figure shows very good matching between the two results. The lateral and axial motions of the 3D model, not shown, are very low.

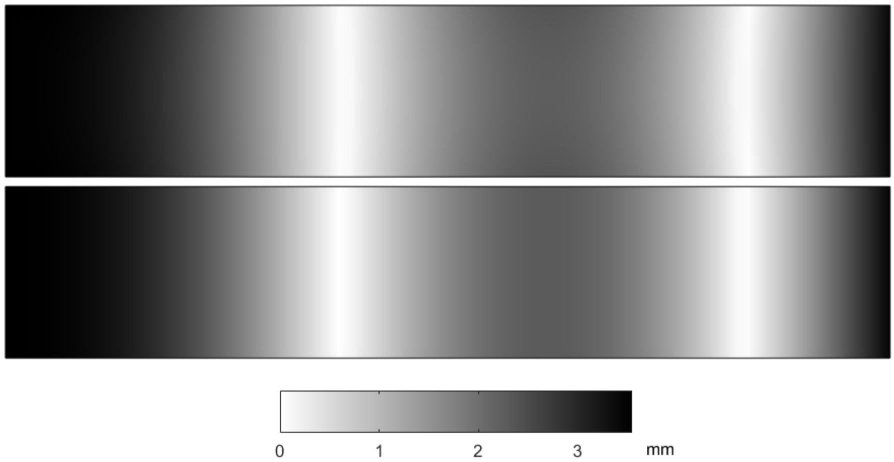

Figure 3 – Top view of the RMS displacement of the beam driven by vertical motion of its left end. Top: Euler-Bernoulli 1D model; bottom: boundary forming 3D approach.

3.3 Response to mixed excitation

The next example shows the RMS displacement of the beam excited at the left end as in the previous example and, in addition, by a vertical force of 200kN centered laterally at 0.37m from the driven end. Good matching with the result obtained by the Euler-Bernoulli model is again noticeable. A slight shift in the position of the minimum displacement on the left side is visible between the two results. Since neither of the two models is fully accurate it is difficult to identify offhand the reason for this discrepancy.

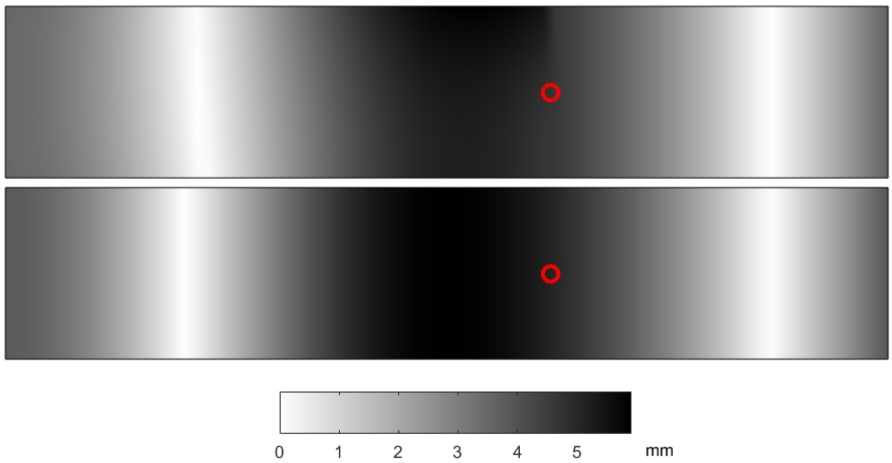

Figure 4 – Top view of the RMS displacement of the test beam driven by vertical displacement at its left end and a vertical force. Top: Euler-Bernoulli 1D model; bottom: boundary forming 3D approach.

In the next example the full diversity of boundary conditions will be applied: the left end of the beam is driven by a prescribed displacement, the right end is clamped while two unequal forces act at the longitudinal centerline of the beam. The two forces located at 0.37m and 0.44m, shifted in phase by 90°, have the amplitudes of 200kN and 300kN respectively.

The rectangular shape is a very difficult one to handle by the present approach in view of abruptly changing state of stress in the vicinity of corners. In spite of the difficult boundary conditions the matching of the two solutions remains very good.

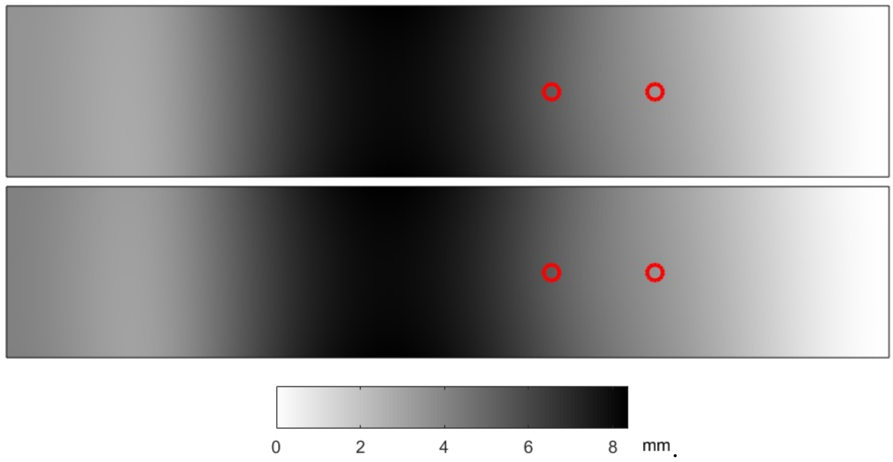

Figure 5 – Top view of the RMS displacement of the test beam driven by vertical displacement at its left end and two vertical forces. Top: Euler-Bernoulli 1D model; bottom: boundary forming 3D approach.

4. CONCLUSIONS

The boundary forming approach, previously validated for bodies with entirely free boundaries, has been validated in this paper for bodies of mixed excitation and boundary conditions. One part of the boundary can be driven by imposed motions while the rest can be partially free and partially restrained against motion. External forces acting on the body can be added indiscriminately.

The approach is based on the analytical solution to point force excitation of an infinite elastic solid which is used to reconstruct the boundary conditions of the targeted object. The validation was done by comparing the results of forced vibration obtained in two ways: by a 3D model of a slender beam and by the classical Euler-Bernoulli model of the same beam vibrating in flexure. Good matching was obtained at a sufficiently low frequency at which the classical model provides satisfactory results. Since a beam of sharp edges is particularly difficult to model with the developed approach it can be expected that the bodies of oval shape with gradually changing geometry can be modelled with an increased accuracy and simplicity.

The selection of model parameters is a critical step of the present method. It represents a topic on its own and largely exceeds the scope of this paper the purpose of which is to provide the outline of the boundary forming approach applied to vibration with mixed excitation and boundary conditions.

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