The use of a serial system of cylindrical and concentric Helmholtz resonators for Low Frequency Sound Absorption

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Abstract
A theoretical and numerical study of sound propagation in a serial system of cylindrical and concentric Helmholtz resonators is conducted. By altering the geometry of each subsequent Helmholtz resonator it is possible to generate very low frequency resonances. Using the transfer matrix method an analytical formulation for the acoustic absorptive properties of the this serial configuration is proposed. An optimization method is developed to tune the system for the absorption of sound waves at specific frequency ranges. Through this study it is shown that using an optimized configuration of Helmholtz resonators in the low frequency range, sound absorption can be achieved at sub-wavelength sample thicknesses. The results of the analytical study are validated through numerical means.
Keywords: Helmholtz resonator, Absorption, Optimisation

1 INTRODUCTION

Typically used sound absorbers consist of bulky porous/ fibrous materials which is often impractical for low frequency sound absorption due to the mass-density law [2]. To overcome this, acoustic metamaterials consisting of Helmholtz resonators are have been utilised to achieve broadband sound absorption at sub-wavelength thicknesses [3]. Of particular relevance to this paper is the extensive research that has been undertaken on cylindrical and concentric Helmholtz resonators using the Transfer Matrix Method (TMM) [5, 9, 7, 6], highlighting how this relatively simple analytical method can be a useful tool for the analysis and optimisation of various configurations of Helmholtz resonators.

This report sets out a theoretical model to analyse the absorption characteristics for a serial system of N cylindrical, concentric Helmholtz resonators, extending the work conducted on duel resonators in [10]. This also provides a simple and computationally inexpensive method in which optimisation can then be implemented to maximise the absorption at desired frequencies. A comparison is then made between the results obtained analytically and those produced by numerical analysis for validation. This is done for the case of three serial Helmholtz resonators.

2 GEOMETRY

An axisymmetric view of the studied geometry can be seen in figure 1. Here it is evidenced that the geometry consists of a waveguide in which the end is loaded with a serial system of Helmholtz resonators. These are placed end-to-end and the Nth resonator cavity results in velocity termination. The plane wave propagates from left to right. It is worth noting that Rt denotes the radius of the waveguide, Rn_i denotes the neck radius for the ith HR and Rc_i denotes the cavity radius for the ith HR. Ln_i and Lc_i denote the length of the neck and cavity for the ith HR, respectively.

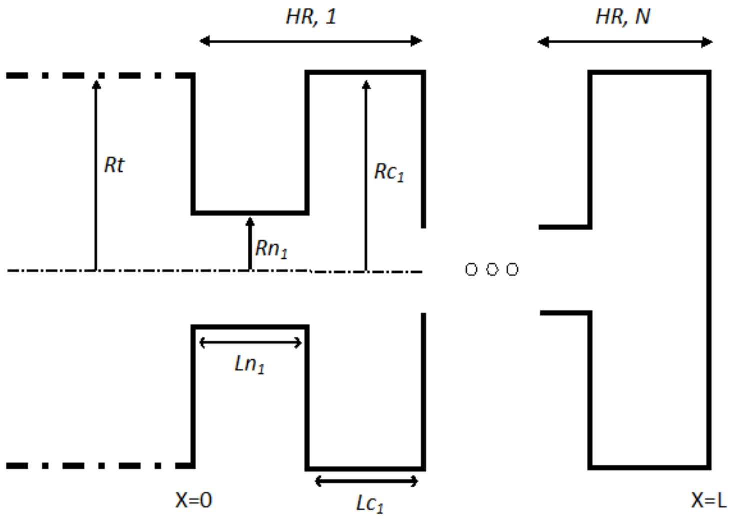

Figure 1. Geometry for seriel system of N Helmholtz resonators

3 THEORETICAL MODEL

3.1 Transfer Matrix Method

The theoretical modelling is performed by using the transfer matrix method, which relates the initial sound pressure, p, and normal acoustic particle velocity, v, at the start (x=0) and end (x=L) of the system of Helmholtz resonators. The transfer matrix, T, is derived under the assumption that only plane waves propagate in the system. From this the reflection and transmission coefficients can be calculated. The system is written as

$$\begin{bmatrix} p \\ v \end{bmatrix}_{x=o} = T \begin{bmatrix} p \\ v \end{bmatrix}_{x=L} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix}_{x=L} \tag{1}$$

Where T is the product of the transfer matrices for each section of each HR within the system,

$$T = \prod_{i=1}^{N} M_n^i M_{\Delta l}^i M_c^i \tag{2}$$

The transmission matrix for the ith HR neck and cavity take the following forms, respectively.

$$M_n^i = \begin{bmatrix} cos(k_n^i Ln_i) & iZ_n^i sin(k_n^i Ln_i) \\ \frac{i}{Z_n^i} sin(k_n^i Ln_i) & cos(k_n^i Ln_i) \end{bmatrix} \tag{3}$$

$$M_c^i = \begin{bmatrix} cos(k_c^i Lc_i) & iZ_c^i sin(k_c^i Lc_i) \\ \frac{i}{Z_c^i} sin(k_c^i Lc_i) & cos(k_c^i Lc_i) \end{bmatrix} \tag{4}$$

Where $k_{n,c}^i = \omega \sqrt{\frac{\rho_{n,c}^i}{\kappa_{n,c}^i}}$ and $Z_{n,c}^i = \frac{1}{S_{n,c}^i} \sqrt{\kappa_{n,c}^i \rho_{n,c}^i}$ are the effective wavenumber and characteristic impedance of the ith HR neck or cavity, $\kappa_{n,c}^i$ and $\rho_{n,c}^i$ are the effective bulk modulus and density, respectively, and $S_{n,c}^i$ is the cross sectional area.

The transmission matrix that accounts for the end corrections of the neck of the ith HR is written as

$$M_{\Delta l}^{i} = \begin{bmatrix} 1 & iZ_n^i k_n^i \Delta l \\ 0 & 1 \end{bmatrix} \tag{5}$$

Where Δl is arrived at from the addition of two correction lengths $\Delta l = \Delta l_1 + \Delta l_2$ written as

$$\Delta l_1 = 0.82\left[1 - 1.35\frac{r_n^i}{r_c^i} + 0.31\left(\frac{r_n^i}{r_c^i}\right)^3\right] r_n^i \tag{6}$$

$$\Delta l_2 = 0.6 r_n^i \tag{7}$$

The first length correction, Δl_1, is due to pressure radiation at the discontinuity from the neck to the cavity of the HR [4], while the second length correction, Δl_2, comes from the radiation at the discontinuity from the neck to the surrounding medium [1].

To determine the entrance impedance for a system of resonators you multiply the final T matrix by $[1,0]^T$ which accounts for the velocity termination at the end of the final resonator. From this the impedance can simply be found as follows

$$Z = \frac{P_{x=0}}{v_{x=o}} = \frac{T_{11}}{T_{21}} \tag{8}$$

Similarly, the reflection coefficient is also determined by the final Transmission matrix and can be found using the following expression

$$R = \frac{T_{11} - T_{21} Z_0}{T_{11} + T_{21} Z_0} \tag{9}$$

from which the absorption coefficient can be computed by

$$\alpha = 1 - |R|^2 \tag{10}$$

3.2 Visco-thermal Losses

The propagation of acoustic plane waves in a circular cross section of radius r, with viscothermal losses, is given by the following complex frequency dependant density and bulk modulus, as described in [8].

$$\rho = \rho_0\left(1 - \frac{2J_1(rG_r)}{rG_r J_0(rG_r)}\right) \tag{11}$$

$$\kappa = K_0\left(1 + (\gamma - 1)\frac{2J_1(rG_k)}{rG_k J_0(rG_k)}\right) \tag{12}$$

Where $G_r = \sqrt{\frac{-i\omega\rho_0}{\eta}}$ and $G_k = \sqrt{\frac{-i\omega\rho_0 Pr}{\eta}}$, in which ρ_0 is the density, $K_0 = \gamma P_0$ is the bulk modulus, γ is the ratio of specific heats, P_0 is the atmospheric pressure, Pr is the Prandtl number and η is the dynamic viscosity.

4 Optimisation

In order to achieve maximum absorption for specified frequencies, whilst adhering to geometrical constraints, a method of optimisation was used to output the optimal dimensions required. This was implemented by first creating a square wave with a value of 1 around the target frequencies. Then by limiting the absorption coefficient function to the frequencies in which the square wave is 1, the optimal dimensions are outputted by minimising the following objective function.

$$Objective\ function = area(square\ wave)\ -\ area(\alpha) \tag{13}$$

Using this method for target frequencies of 100Hz, 300Hz and 500Hz with a maximum radius of 50mm for the waveguide and cavities (noting that the radius of the necks is always less) the following optimised absorption coefficient for three nested Helmholtz resonators is produced. The total length of this system is 7.05cm which equates to a sample thickness ratio of $\lambda/49$ at 100Hz, where an absorption of 0.77 is achieved.

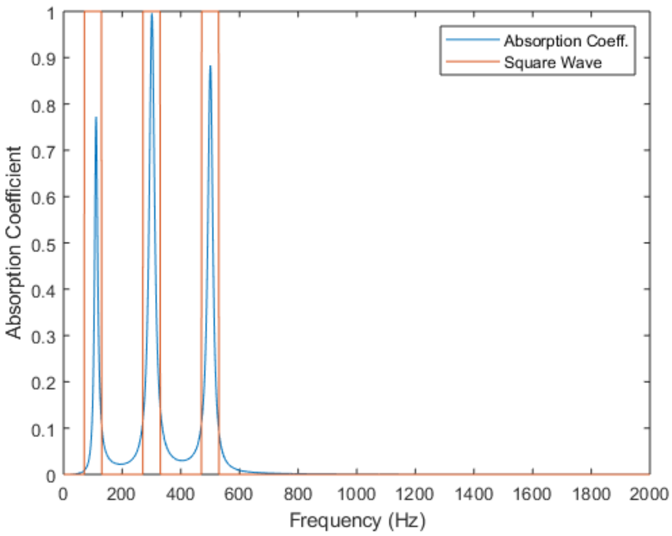

Figure 2. Optimised absorption coefficient for three Helmholtz resonators in series

It is worth noting that each subsequent resonator in the series is resonating at a lower frequency than the last. This is achieved, in part, by the reduction in neck radius of each subsequent resonator. This is evidenced in table 1 which contains the optimised geometry for this specific case.

Table 1. Optimised geometry dimensions for serial array of three Helmholtz resonators

N	1	2	3
Rn (mm)	6.40	2.90	1.20
Rc (mm)	50.0	45.6	49.6
Ln (mm)	5.40	4.50	2.2
Lc (mm)	22.0	15.6	20.8

5 Numerical Validation

In order to provide numerical validation for the analytical method a model was produced in COMSOL multiphysics 5.0 using the dimensions obtained in the optimisation process. This utilises the finite element method (FEM) to calculate for the complex pressure within a specified geometry. The modelling of viscothermal losses was achieved by using the 'circular duct' model within the 'narrow regions' feature of the acoustic module within COMSOL. This was applied specifically to the neck regions of each HR. The absorption coefficient was ascertained using the two microphone method. The absorption coefficient for both the numerical and analytical methods can be seen in figure 3.

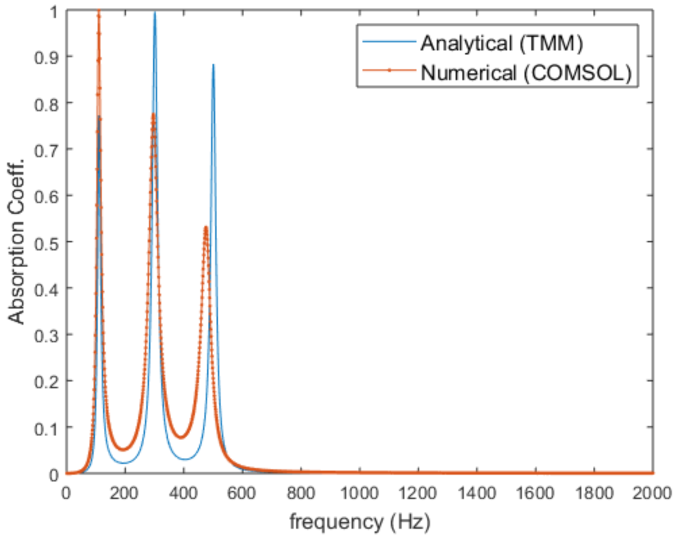

Figure 3. Numerical validation for the optimised absorption coefficient for three Helmholtz resonators in series

From figure 3 it is evident that there is excellent agreement in regards to the resonant frequency of each Helmholtz resonator. However, there is evidently a discrepancy in regards to the amplitude of the absorption peaks. By comparing the numerical results to the analytical it can be seen that the low frequency resonator is over-performing, achieving perfect absorption, with the latter resonators underperforming. Whilst this is a positive results in regards to low frequency absorption it also highlights that the modelling of visco-thermal losses is not completely consistent between the two methods. Regardless of this fact, the analytical model is still validated but further experimental work would be required to fully determine how the interplay of losses affects the absorption peaks.

6 Conclusions and Future Work

In this paper, a simple and effective analytical method of modelling a system of N cylindrical, concentric nested Helmholtz resonators is outlined. Upon implementation into an optimization algorithm to maximise the absorption at specific frequencies given geometrical constraints, an analytical absorption coefficient of approximately 0.77 at 100Hz is obtained. With the overall thickness of the system being 7.05cm the 100Hz absorption occurs at a sample thickness ratio of $\lambda/49$. Further large absorption peaks are obtained at the desired frequencies of 300Hz and 500Hz. This shows it is possible to achieved low frequency sound absorption at sub-wavelength sample thicknesses using this configuration.

Numerical validation highlights excellent agreement on the resonant frequencies of each resonator, proving the TMM matrix accounts for the coupling that occurs between resonators. However, there is some disagreement in the amplitude of the absorption peaks which points to minor inconsistencies in the modelling of the visco-thermal losses. In the numerical model perfect absorption is obtained at 100Hz (an over-performance) and the two higher frequency resonators under-perform in regards to absorption. Overall the model is still valid but further understanding of the losses is still required.

Further work needs to be conducted in providing experimental validation to the results attained here to solidify the theory as correct. Additionally, more work can be done on understanding the mechanism behind the coupling of Helmholtz resonators in this configuration, which could prove useful for low frequency sound absorption. Also, a larger system of resonators (3+) can be analysed and porous media introduced which could lead to more broadband low frequency absorption instead of the individual resonator peaks seen here.

REFERENCES

[1] V. Dubos, J. Kergomard, A. Khettabi, J.-P. Dalmont, D. Keefe, and C. Nederveen. Theory of sound propagation in a duct with a branched tube using modal decomposition. *Acta Acustica united with Acustica*, 85:153–169, 03 1999.

[2] N. A. Jean F. Allard. *Propagation of Sound in Porous Media*. John Wiley & Sons, 2009.

[3] N. Jiménez, V. Romero-García, V. Pagneux, and J.-P. Groby. Rainbow-trapping absorbers: Broadband, perfect and asymmetric sound absorption by subwavelength panels for transmission problems. *Scientific Reports*, 7(1):13595, Oct. 2017.

[4] J. Kergomard and A. Garcia. Simple discontinuities in acoustic waveguides at low frequencies: Critical analysis and formulae. *Journal of Sound and Vibration*, 114(3):465–479, may 1987.

[5] O. Richoux and V. Pagneux. Acoustic characterization of the hofstadter butterfly with resonant scatterers. *Europhysics Letters (EPL)*, 59(1):34–40, jul 2002.

[6] V. Romero-García, G. Theocharis, O. Richoux, and V. Pagneux. Use of complex frequency plane to design broadband and sub-wavelength absorbers. *The Journal of the Acoustical Society of America*, 139(6):3395–3403, jun 2016.

[7] S.-H. Seo and Y.-H. Kim. Silencer design by using array resonators for low-frequency band noise reduction. *The Journal of the Acoustical Society of America*, 118(4):2332–2338, oct 2005.

[8] M. R. Stinson. The propagation of plane sound waves in narrow and wide circular tubes, and generalization to uniform tubes of arbitrary cross-sectional shape. *The Journal of the Acoustical Society of America*, 89(2):550–558, feb 1991.

[9] G. Theocharis, O. Richoux, V. R. García, A. Merkel, and V. Tournat. Limits of slow sound propagation and transparency in lossy, locally resonant periodic structures. *New Journal of Physics*, 16(9):093017, sep 2014.

[10] M. Xu, A. Selamet, and H. Kim. Dual helmholtz resonator. *Applied Acoustics*, 71(9):822–829, sep 2010.