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A new local boundary integral equation method for meshless acoustic computations

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ABSTRACT

The local boundary integral equation (LBIE) method has been introduced approximately three-decades ago, and has been successfully used in engineering problems such as elasticity, acoustics and electrodynamics. The method is based on writing the local weak form of the governing equation. If an appropriate form of the Green's function can be found considering the governing equation in subdomains, the local integral equation may have a convenient form, e.g. without domain integrals or the flux term on the subdomain boundaries. Considering the application of the methods to acoustic problems, four different test functions have been employed so far in the literature. All of the resulting formulations contain domain integrals. In this paper, we present a new form of Green's function to be used in LBIE method for acoustic calculations, which yields a simple form of the local integral equation without domain integrals. This feature provides significant improvement in terms of the computational time (CPU) requirements. The efficiency and the accuracy of the new method are presented and compared with a Finite Element Method.

Keywords: Meshless methods, local boundary integral

1. INTRODUCTION

The local boundary integral equation (LBIE) method [1, 2, 3, 4, 5] has been introduced about three decades ago as a robust numerical method for engineering problems; it is one of the version of the meshless local Petrov-Galerkin method [6]. The method is based on writing the local weak form in spherical subdomains (which may be overlapping) and applying the divergence theorem. The resultant local boundary and/or domain integrals are numerically discretized. The field variable needs to be approximated at the numerical integration points using some kind of interpolation technique. The Moving Least Squares (MLS) approximation has been employed for this purpose in majority of the studies [1, 2, 3, 4, 5, 6, 7, 8], whereas the Radial Basis Function (RBF) interpolation technique has also been used in some other works [9, 10, 11, 12, 13, 14].

The MLPG and LBIE methods have been applied to wave propagation problems in Refs. [2, 3, 4, 5, 7, 8, 9, 10, 11, 12], in order to solve for the Helmholtz equation. In Ref. [7] the Gaussian weight function was used as the test function. Ref. [8] used the Heaviside step function as the test function. Refs. [3, 2, 4, 9, 10] employed the modified fundamental solution of the Laplace equation as the test function to the Helmholtz equation. Refs. [5, 11, 12] applied the modified fundamental solution of the Helmholtz equation as the test function. The final form of the local integrals in all of these studies [2-5, 7-12] contain domain integrals.

In Ref. [15], we have reconsidered the derivation of an appropriate test function to be used in the weak form in meshless methods. Explicitly, we have derived the unique solution of the Dirichlet's boundary value problem in the local spherical subdomain with zero boundary condition value. The derived function is a special form of the Green's function (tailored Green's function) which satisfies the objective of eliminating the domain integrals and the flux term from the local integral equation. As a result, a new form of the LBIE method has been established, which has improved features compared to the ones mentioned above.

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2. FORMULATIONS

We consider the Helmholtz equation

$$\nabla^2 p(\boldsymbol{x}) + k^2 p(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in V, \tag{1}$$

where p is the acoustic pressure and k is the wavenumber. The domain V is enclosed by $\Gamma = \Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{q}}$ with the boundary conditions

$$p(\boldsymbol{x}) = p_0, \qquad \text{on} \qquad \Gamma_{\mathrm{u}}, \tag{2}$$
$$\frac{\partial p(\boldsymbol{x})}{\partial n} \equiv q = q_0, \qquad \text{on} \qquad \Gamma_{\mathrm{q}},$$

where p_0 is the prescribed acoustic pressure on the essential boundary Γ_{u}, q_0 is the prescribed potential flux on the boundary Γ_{q}, and n is the outward normal vector on Γ.

In the traditional boundary element method (BEM), a weak formulation of the problem is formed in the global solution domain V using the free space Green's function. In the LBIE, it is assumed that the governing equation holds for any local subdomain and a set of N spherical *subdomains* are distributed in the domain V and on the boundary Γ. The centers of the subdomains are called as *source nodes* where the volumes in $\mathcal{R}^3$ for each node are denoted as Ω, such that $\forall\Omega \subset V$. The radius of each subdomain is denoted as R_Ω. The surface (boundary) enclosing each interior subdomain is defined as $\Omega_{\mathrm{S}} = \partial\Omega$. When the source node lies on the global boundary Γ, the subdomain does not take the shape of a full sphere anymore. Instead, a half sphere or a slice of a sphere is formed depending on the shape of the global boundary. For such cases, we set Ω_{b} to denote the part of the subdomain boundary which intersects the global boundary Γ. For instance, for the subdomains which lie on Γ, we have $\partial\Omega = \Omega_{\mathrm{b}} \cup \Omega_{\mathrm{S}}$, where $\Omega_{\mathrm{b}} \subset \Gamma$ and $\Omega_{\mathrm{S}} \subset V$ (Fig. 1). The following weak formulation can be written for *each* local subdomain.

$$\int_{\Omega} G\ \ (\boldsymbol{x},\boldsymbol{y})\ \ (\nabla^2 p\ \ (\boldsymbol{x}) + k^2 p(\boldsymbol{x}))\, d\Omega = 0. \tag{3}$$

An appropriate test function G should be used for Eq. (3). As mentioned in the Introduction, the previous meshless methods have employed different types of test functions, such as the Heaviside step function, the Gaussian weight function, the modified fundamental solution of the Laplace equation or the modified fundamental solution of the Helmholtz equation. In Ref. [15], a new test function has been derived which is the unique solution of the Dirichlet's problem for a spherical subdomain with the boundary value zero. The derived test function is given explicitly as:

$$G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\frac{\sin k(R_\Omega - |\boldsymbol{x}-\boldsymbol{y}|)}{\sin kR_\Omega}\ . \tag{4}$$

Using the tailored Green's function (4) in Eq. (3), and applying integration by parts, we obtain the integral

$$p(\boldsymbol{y}) - \frac{k}{4\pi R_\Omega \sin kR_\Omega}\int_{\partial\Omega} p(\boldsymbol{x})\, d\partial\Omega(\boldsymbol{x}) = 0 \tag{5}$$

for local subdomains which lie fully inside the domain.

When the source point $\boldsymbol{y}$ lies on the global boundary, the subdomains are taken as a part of the sphere (e.g. half sphere). In this case, Ω_{s} denotes the boundary which lies within the global volume V, Ω_{b} indicates the boundary which intersects with Γ. The source point (center of the local subdomain) lies on Ω_{b}, i.e. $\boldsymbol{y} \in \Omega_{\mathrm{b}} \subset \Gamma$. The local integral equation for the boundary nodes takes the form:

$$\lambda\ p(\boldsymbol{y}) - \frac{k}{4\pi R_\Omega \sin kR_\Omega}\int_{\Omega_{\mathrm{S}}} p(\boldsymbol{x})\, d\partial\Omega_{\mathrm{S}} = \int_{\Omega_{\mathrm{b}}} G(\boldsymbol{x},\boldsymbol{y})\frac{\partial p(\boldsymbol{x})}{\partial n}\, d\partial\Omega_{\mathrm{b}}, \tag{6}$$

where λ is a parameter analogous to the geometric coefficient in the BEM, which is given here as

$$\lambda(\boldsymbol{y}) = -\frac{1}{4\pi}\int_{\Omega_{\mathrm{S}}} \frac{\partial r}{\partial n(\boldsymbol{x},\boldsymbol{y})}\frac{\partial}{\partial r}\ \left(\frac{1}{r}\right) d\partial\Omega_{\mathrm{S}}. \tag{7}$$

Eqns. (5) and (6) need to be numerically calculated. This requires, first, the discretization of the boundary integrals in these equations and then the interpolation of the unknown p at the discretization points. Unlike the boundary element method, there is no need to use elements to discretize the surface integral in the LBIE method. The details of the quadrature and interpolation methods in the current meshless method can be found in detail in Ref. [15].

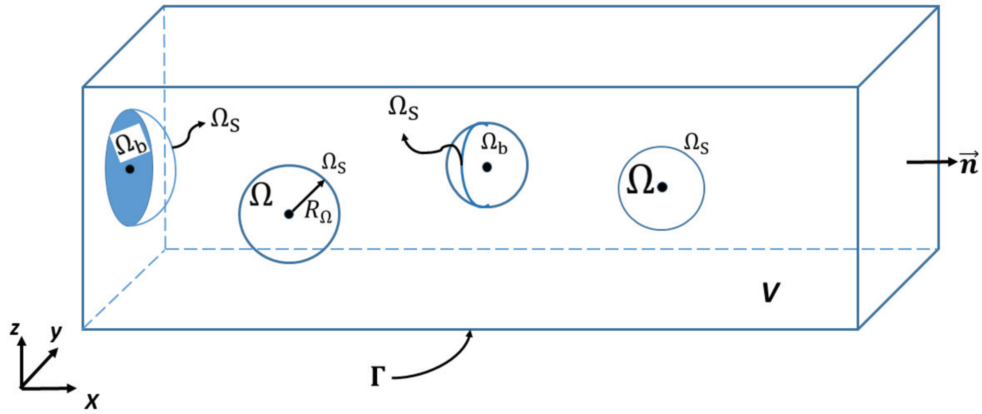

Figure 1 – The sketch of the spherical local subdomains Ω, and the enclosing surfaces Ω_S. For the subdomains on the global boundary, Ω_b defines the surface which intersects with the global boundary Γ (the filled blue area).

3. NUMERICAL EXAMPLES

In this section, the numerical examples will be shown in order to demonstrate the efficiency and the accuracy of the new method. The problem domain is chosen as a rectangular cuboid filled with air. The density of air is taken as $\rho = 1.3$ kg/m^3 and the speed of sound in air is taken as $c = 340$ m/s.

In the simulations, the nodes are distributed uniformly in the domain with spacing h between two adjacent nodes. The boundary integrals were calculated numerically using $N_\text{RBF} = 50$ neighbor nodes in the radial basis function interpolation and $N_G = 10$ quadrature points (in each dimension). The computations were performed with MATLAB software on a desktop workstation computer with 3.20 GHz processor speed and 128 GB of RAM. The resultant system matrices were solved iteratively using the *gmres* function. The accuracy of the numerical results is assessed by evaluating the L_1 norm error

$$e_{L_1}(x) = \frac{|p(\boldsymbol{x}) - p_\text{exact}(\boldsymbol{x})|}{|p_\text{exact}(\boldsymbol{x})|}, \tag{9}$$

and the L_2 norm error

$$e_{L_2} = \sqrt{\frac{\sum_{i=1}^{N}\left[p_\text{numerical}^{(i)} - p_\text{exact}^{(i)}\right]^2}{\sum_{i=1}^{N}\left[p_\text{exact}^{(i)}\right]^2}}. \tag{10}$$

3.1 Standing waves in a duct

We consider a domain with length L=0.4m, width w=0.15m, and height H=0.06m. The inlet and the outlet of the rectangular duct is located on y-z plane, at x=0 and at x=L, respectively. The remaining boundaries, which are on the x-y and x-z planes, are rigid walls. In particular, a constant pressure is prescribed at the inlet, i.e.

$$p_\text{i}|_{x=0} = 1000 \;\; Pa, \tag{11}$$

and, a full-reflection condition is given at the outlet:

$$\left.\frac{\partial p}{\partial n}\right|_{x=L} = 0. \tag{12}$$

For the conditions given above, the problem reduces to 1D wave propagation in x-direction. One can show that the exact solution to the problem defined in Eqns. (1), (11) and (12) is given by

$$p_{\text{exact}}(x) = p_{\text{i}} \frac{\cos k(x-L)}{\cos kL}. \tag{13}$$

The computational time and the accuracy of the method are presented in Table 1. The frequency is set as f=5000 Hz. The second column in Table 1 shows the total number of nodes N for the corresponding value of h, hence the degrees of freedom (DOF) being solved for. The 3rd column indicates the CPU time required for the preparation of the matrix coefficients for the nodes on the boundary and in the domain. The 4th column presents the total CPU time including the preparation of the system matrix and the iterative solver. The results are compared to the formulation in Ref. [12]. Note that the method in [12] in the Mach number limit $M \to 0$ becomes the version of the LBIE which contains domain integral. It can be seen that the accuracy of the present method and Ref. [12] are comparable and show only minor difference, which may be because of the different integration procedures. However, the computational time requirement is significantly improved with the new method. For a Gaussian quadrature integration with N_G weights, a domain integration needs $\sim N_G^3$ operations, whereas a surface integration is performed with $\sim N_G^2$ operations. Therefore, a time improvement with factor $1/N_G$ is achieved. This can be easily noticed by comparing the time required for the preparation of the system matrix, i.e. the CPU time for the assembly. The CPU time for solving the matrix is also significantly improved as can be seen in Table 1.

Table 1 – CPU time and L_1-norm error for the case f =5000 Hz

		Current Method			Ref. [12]		
h (mm)	**Nodes (*N*)**	**CPU Time Assembly (sec)**	**CPU Time Total (sec)**	**L_1 Error e_{L_1} (%)**	**CPU Time Assembly (sec)**	**CPU Time Total (sec)**	**L_1 Error e_{L_1} (%)**
10	9594	5.4	20.4	1.36 e-3	49	80	9.89 e-4
8	17952	6.6	29	1.06 e-3	59	104	9.44 e-4
5	70875	11	68	5.53 e-4	102	226	5.36 e-4
2.5	544341	25	2172	8.71 e-5	205	4732	7.49 e-5

3.2 Comparison with COMSOL FEM

The accuracy of the method is further presented in Table 2, and compared with Ref. [12] and with the COMSOL FEM software. A uniform mesh with h=5 is created for the meshless methods. Similarly, quadratic cubic elements with 5 mm side length are used in FEM computations, which result in 70875 domain elements (see Fig. 2a). Although equal number of nodes are used both in meshless methods and in FEM, the DOF being solved for is much larger (~5.5 x 10^5) in the latter. An important dimensionless parameter to assess the resolution of the mesh for wave propagation problems is λ/h, which is reported in the second column. The L_2-norm errors decrease consistently with increasing DOF for all formulations, indicating that the methods are convergent. However, the meshless methods are substantially more accurate. Although not presented in Table 2, the CPU time requirements of the meshless method is comparable to the FEM solution. The total CPU time spent in the present method is 68 seconds, whereas the FEM simulation takes 42 seconds. Though, one should note that the mesh discretization, the mesh connectivity and the system matrix assembly are generally done as pre-processing operations in the FEM software. As an example, the wave propagation result at 5000 Hz using the FEM is presented in Fig. 2b.

Table 2 – L_2-norm error for dx =5 mm.

		Current Method	**Ref. [12]**	**COMSOL**
Freq. (*Hz*)	**λ/h**	e_{L_2}	e_{L_2}	e_{L_2}
15000	4.57	0.0046	0.0047	0.01694
10000	6.86	0.001	0.0011	0.1128
5000	13.72	9.67 e-4	9.50 e-4	0.0438
2000	34.29	5.66 e-4	1.83 e-4	0.0160
1000	68.60	1.20 e-4	1.09 e-5	0.0073

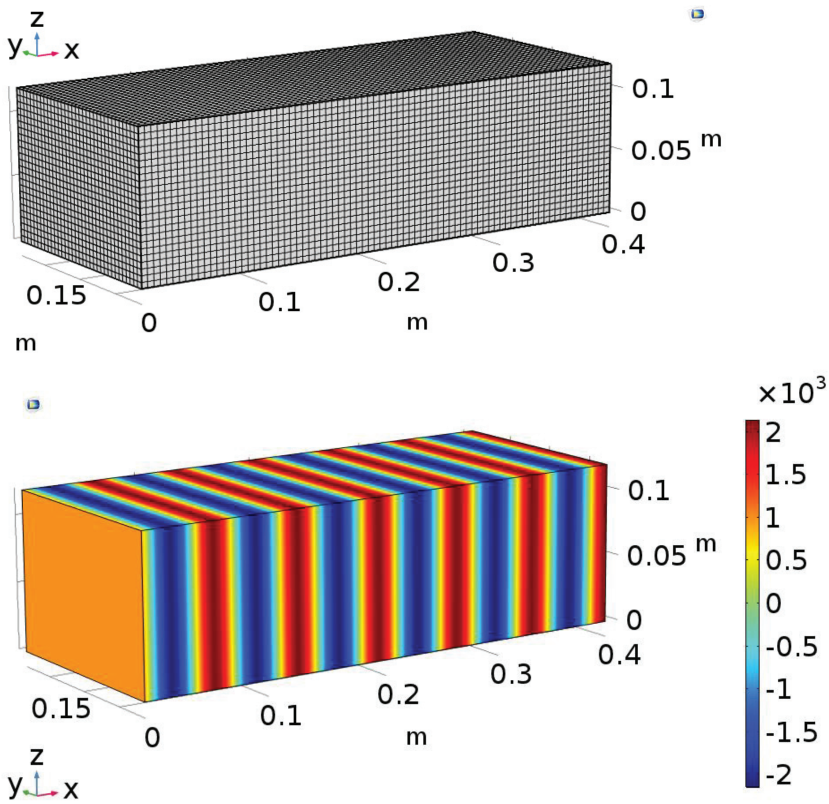

Figure 2 – a) Uniform FEM mesh with quadratic elements with side length 5 mm. b) FEM solution for the real part of acoustic pressure at f=5000 Hz

4. CONCLUSIONS

In this paper, a new LBIE method for the solution of the Helmholtz equation is presented. Unlike the previous formulations in this category, the present method does not contain any domain integral or the flux term. In the final form of the local integral, the acoustic pressure is the only unknown which needs to be approximated by any kind of mathematical interpolation. The computational time requirements of the method have shown to be $\sim 1/N_G$ times of that encountered in the predecessors of the method. Moreover, the method yields a real-valued, band-limited system matrix. Provided that more efficient linear solvers appropriate for these features of the system matrix are used, the method can be used as a better alternative for the existing numerical FEM and BEM techniques. The comparison of the method with an FEM software (COMSOL v4.3 with quadratic elements) for identical discretization of the problem domain reveals at least one order of magnitude more accurate results for the meshless method. The proposed new LBIE method can be applied to many other types of wave propagation problems, provided that circular local subdomains are used in the implementation.

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