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Physical principle of pitch bent by cross-fingering

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Abstract

Cross-fingering is a technique of playing woodwind instruments in which one or more tone holes are closed below the first open hole. It usually yields a pitch lower than that played with normal fingering. However, pitch is raised in exceptional cases. Pitch flattening has been traditionally understood using the lattice tone hole theory. On the other hand, pitch sharpening has been scarcely explained except for pointing out the possibility for the open hole to act as a register hole. This paper proposes understanding these pitch bending phenomena in a unified manner with a model of two coupled mechanical oscillators. Bores upstream and downstream of the open hole interact with each other by sharing the air in the open hole oscillating as a lumped mass. This mechanism is known in physics as avoided crossing or frequency repulsion. With an extended model having three degrees of freedom, pitch bending of the recorder played with cross-fingering in the second register can also be explained.

Keywords: woodwind instruments, resonance frequencies, avoided crossing

1 INTRODUCTION

Systematic research on passive acoustic resonance of the woodwind instruments aiming at their tone hole design dates back in the 1960s [1, 2]. In the following decades, where digital computers were dramatically developed, the theories and models obtained in the research were applied to practical design of the instruments while they were developed and refined on their own [3, 4, 5, 6]. From the engineering point of view, it can be said that passive acoustics of the woodwind instruments has attained perfection today. It is possible to design the instruments in a satisfactory manner with the aid of computer analysis based on the theories.

However, it is not always sufficient from the physical point of view to understand the mechanism by which the resonance characteristics of the instrument appear. The acoustics of cross-fingering is one such topic. Pitch flattening due to cross-fingering has been traditionally understood using the lattice tone hole theory [7]. On the other hand, pitch sharpening has been scarcely explained except for pointing out the possibility for the open hole to act as a register hole [8]. Recently, Yoshikawa and Kajiwara [9, 10] shed a new light on this problem by examining pitch bending in a shakuhachi played with cross-fingering experimentally in detail. Adachi [11] proposed a minimal model with which both pitch flattening and sharpening due to cross-fingering can be understood in a unified manner. This paper attempts to explain the pitch bending mechanism based on this model as plainly as possible.

Table 1 lists a few examples of cross-fingerings on the alt recorder. The first three examples are at the first register and the last four are at the second register. The pitch is lowered in the first five examples, whereas it is raised in the last two. The general tendency is that the pitch is lowered at the lower register. The pitch tends to rise at the higher register and when more holes are closed. Note that the pitch sharpening also occurs at the first register, for example, when two more downstream open tone holes of fingering C♯5 are closed, although this fingering is not used in a normal performance.

2 RESONANCE CHARACTERISTICS OF A ONE-HOLE FLUTE

Pitch bending with cross-fingering is due to the interaction between the two parts of the instrument bore upstream and downstream of the open hole. To overview this interaction, a simplified flute having only one open tone hole is considered first. Figure 1 (a) shows the model of this flute. The section from the input (left) end to the open hole is called upper bore and that from the hole to the output (right) end is called lower bore. The

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Table 1. A few examples of cross-fingerings on the alt recorder. The symbols denote ●: open, ○: close and ◑: 'pinched' (half open).

Normal fingering			Cross-fingering	
C5:	● ●●● ○○○○	pitch lowered ⟶	B4:	● ●●● ○●●○
D5:	● ●●○ ○○○○	pitch lowered ⟶	C♯5:	● ●●○ ●●○○
E5:	● ●○○ ○○○○	pitch lowered ⟶	E♭5:	● ●○● ●○○○
C6:	◑ ●●● ○○○○	pitch lowered ⟶	B5:	◑ ●●● ○●○○
D6:	◑ ●●○ ○○○○	pitch lowered ⟶	C♯6:	◑ ●●○ ●○○○
D6:	◑ ●●○ ○○○○	**pitch raised** ⟶	E6:	◑ ●●○ ●●○○
D6:	◑ ●●○ ○○○○	**pitch raised** ⟶	D♯6:	◑ ●●○ ●●●○

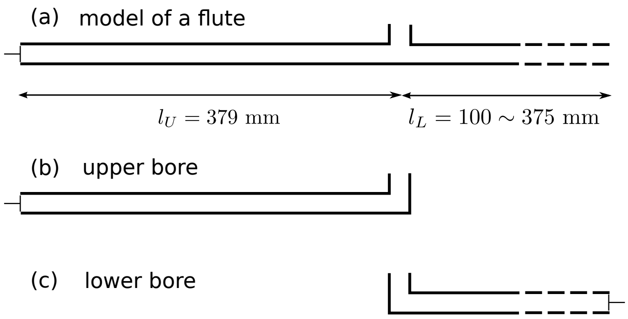


Figure 1. (a) A simplified flute having one open tone hole. (b) The upper part and (c) the lower part of the instrument bore.

upper bore length l_{U} is fixed to 379 mm, while the lower bore length l_{L} is increased from 100 to 250 mm with a 25 mm step. The bore radius is assumed to be 9.5 mm, the tone hole radius to be 9.5 mm and its acoustical length h_{e} to be 22 mm.

The resonance characteristics of the flute are represented by the input admittance, which is the ratio of the volume velocity to the sound pressure at the input (left) end when the flute is excited by a piston as shown in Figure 1 (a). The blue thick curves in Figure 2 show the input admittance calculated using the transmission-line matrix model [12] and the tone hole theory [6]. The flute resonates at the frequencies where the admittance takes maxima and can be played near one of these frequencies. These plots are essentially the same as Figure A6.3 in [2]. To help understanding the admittance of the flute, two additional admittances of the upper and lower bores are plotted with the green and red thin curves, respectively. The upper bore admittance can be conceptually measured in the setup illustrated in Figure 1 (b). The bore is closed just after the open hole so that only the air column in the upper bore vibrates while that in the lower bore (not shown in the figure) stays still. Likewise, the lower bore admittance can be obtained as shown in Figure 1 (c). The lower bore closed just before the open hole is excited from the output end in this case.

Resonances of the upper bore (peaks in the green thin curves) occur at 433, 867, 1303 and 1742 Hz. These of course remain the same even if the lower bore length l_{L} is changed. The first resonance of the lower bore (the lowest peak in the red thin curves) gradually shifts its frequency from 1363 to 622 Hz as l_{L} extends from 100 to 250 mm. If the lower bore frequency is far from one of the upper bore frequencies, the flute resonates approximately at the same frequency as that of the upper bore. On the other hand, if the lower bore frequency approaches one of the upper bore frequencies, for example, at the second register near 860 Hz for the case of $l_{\mathrm{L}} = 175\,\mathrm{mm}$, two resonance frequencies of the flute appear; one is shifted upward and the other is shifted

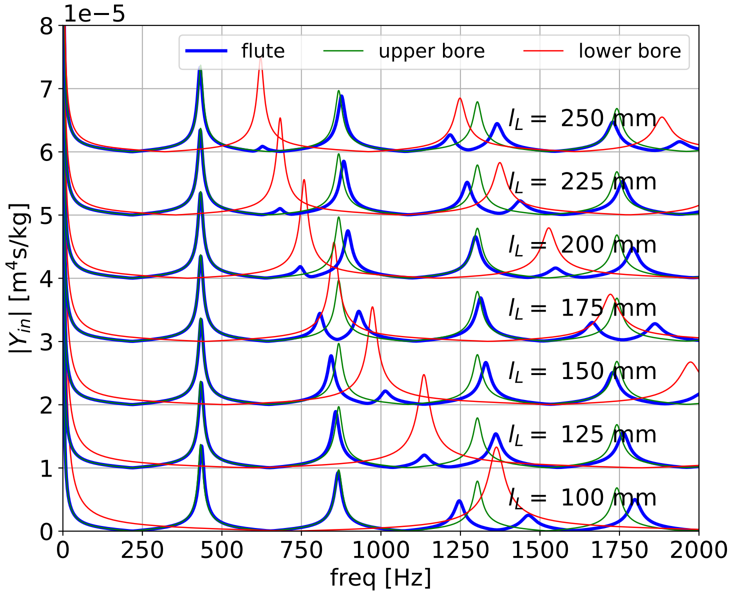


Figure 2. The input admittance spectra of the simplified flute with one open tone hole are plotted in blue thick curves as the lower bore length l_L is increased from 100 to 250 mm while the upper bore length l_U is fixed to 379 mm. The two additional input admittances of the upper and lower bores are also plotted in green and red thin lines to help understanding how the resonance of this flute is generated. The plots are shifted vertically for better visibility.

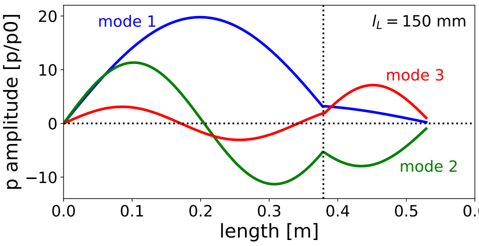
downward from the resonance frequencies of the upper and lower bores. These flute resonance frequencies are formed as if they repel each other even though the frequencies of the upper and lower bores intersect each other. This phenomenon is called avoided-crossing [13, 14]. The avoid crossing happens not only at the second register but also at the third (when $l_L = 100$ mm and 225 mm) and fourth (when $l_L = 175$ mm) registers.

The second resonance frequency f_2 of the flute is apparently lower than the upper bore resonance frequency for $l_L = 150$ and 175 mm. If the flute makes sound at this frequency, the played pitch noticeably becomes lower than the pitch played when l_L is short. On the other hand, the third resonance frequency f_3 is higher than the upper bore resonance frequency for $l_L = 175$ and 200 mm. Sounding at this frequency results in pitch sharpening.

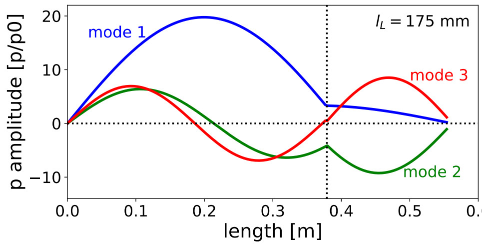
The second and third resonance modes of the flute can be easily discriminated if their standing-wave pressure patterns are compared. For $l_L = 150$, 175 and 200 mm, these (and that of the first mode) are drawn in Figure 3. In the second mode, pressure on both sides of the open hole at $x = 379$ mm vibrates in phase. In the third mode, a pressure node appears near the open hole. On both sides of the node, pressure vibrates in anti-phase.

3 MECHANICAL MODEL OF A ONE-HOLE FLUTE

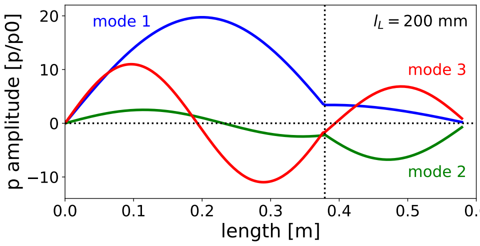
To explain why pitch is bent by cross-fingering, the mechanical model shown in Figure 4 is presented. In this model, the air in the open hole is regarded as mass M vibrating up and down. Two mechanical oscillators, each of which is composed of a spring and a mass, are assigned to the resonances of the upper and lower bores. Masses of the oscillators have the same amount of m. The spring constant of the upper bore is fixed to k, whereas that of the lower bore is varied, with which the lower bore resonance frequency ω_L can be adjusted. As the three masses are linked through lubricating oil at the junction, the two oscillators interact with each other by sharing M.



(a) $l_{\mathrm{L}} = 150$ mm



(b) $l_{\mathrm{L}} = 175$ mm



(c) $l_{\mathrm{L}} = 200$ mm

Figure 3. Standing-wave pressure patterns of the lowest three modes.

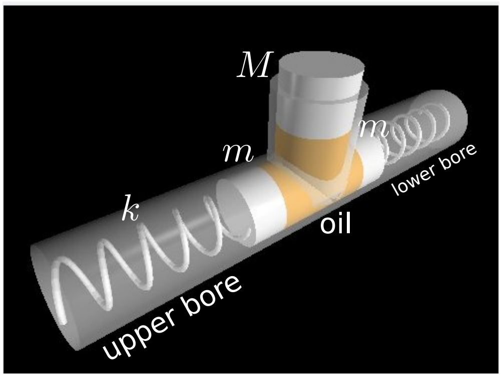
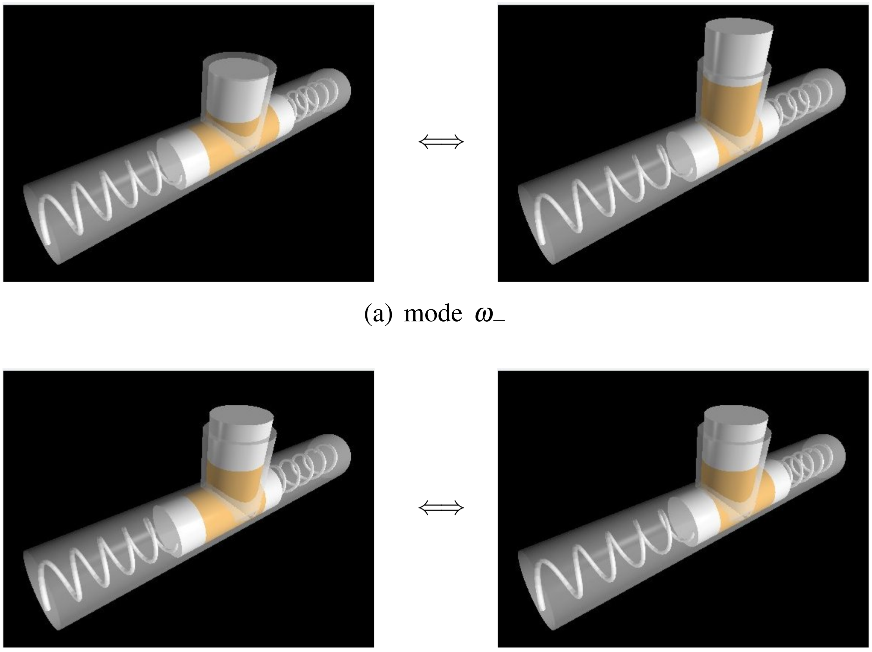


Figure 4. Mechanical model of the one-hole flute.

If the lower bore resonance frequency ω_{L} is much higher than the upper bore resonance frequency ω_{L}, the two bores do not interact. When the upper bore vibrates, mass M vibrates together. However, the lower bore does not vibrate. The vibration frequency in this case is $\omega_{\mathrm{U}} = \sqrt{k/(m+M)}$. This situation correspond to normal fingering. If an actual flute is played with normal fingering, the lower bore or the downstream section of the instrument has a few open tone holes aligned at short intervals. The resonance frequency of this section is much higher than that of the upper bore where the tone holes are closed.

If the lower bore resonance frequency ω_{L} is comparable to the upper bore resonance frequency ω_{U}, the mechanical model has two vibration modes shown in Figure 5. In case (a), the upper and lower bores vibrate symmetrically. Mass M vibrates at twice the acceleration of mass m, so the vibration frequency becomes $\omega_{-} = \sqrt{k/(m+2M)}$, which is lower than $\omega_{\mathrm{U}} \approx \omega_{\mathrm{L}}$. This vibration mode therefore corresponds to the pitch flattening in cross-fingering. In case (b), the upper and lower bore vibrate anti-symmetrically. Mass M does not vibrate. The frequency is then $\omega_{+} = \sqrt{k/m}$, which is higher than $\omega_{\mathrm{U}} \approx \omega_{\mathrm{L}}$. This vibration mode causes pitch



(a) mode ω_{-}

(b) mode ω_{+}

Figure 5. Two vibration modes when $\omega_{\rm U} \approx \omega_{\rm L}$.

sharpening.

As it looks possible to explain the pitch bending due to cross-fingering with this mechanical model, let us write down the equation of motion and calculate $\omega_{\pm}$ in the general case of $\omega_{\rm U} \neq \omega_{\rm L}$. The ratio $r = \omega_{\rm L}/\omega_{\rm U}$ is defined here. Let the displacements of the upper and lower masses m and of the open hole mass M be $x_{\rm U}(t)$, $x_{\rm L}(t)$ and $x(t)$, respectively. Positive $x_{\rm U}(t)$ and $x_{\rm L}(t)$ are defined towards the direction where the springs are stretched. Positive $x(t)$ is defined upward. By disregarding oil inertia, the equation of motion becomes

$$m\ddot{x}_{\rm U} = -kx_{\rm U} - Sp, \quad m\ddot{x}_{\rm L} = -r^2 kx_{\rm L} - Sp, \quad M\ddot{x} = Sp, \tag{1}$$

where S is the cross-sectional area of the bore and $p(t)$ pressure in the junction. If the oil is an incompressible fluid, $x(t) = x_{\rm U}(t) + x_{\rm L}(t)$ is held. The $x(t)$ or $p(t)$ can therefore be eliminated from the equation of motion. When $x_{\rm U}$ and $x_{\rm L}$ vibrate at the same frequency ω, the equation of motion results in

$$\begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \begin{bmatrix} x_{\rm U} \\ x_{\rm L} \end{bmatrix} = \frac{\omega^2}{\omega_{\rm U}^2} \begin{bmatrix} x_{\rm U} \\ x_{\rm L} \end{bmatrix} \tag{2}$$

with $\omega_{\rm U} = \sqrt{k/(m+M)}$ and $\alpha = M/(m+M)$. Solving this equation, we have two eigenfrequencies:

$$\omega_{\pm}^2 = \frac{\omega_U^2 + \omega_L^2 \pm \sqrt{(\omega_U^2 - \omega_L^2)^2 + 4\alpha^2 \omega_U^2 \omega_L^2}}{2(1-\alpha^2)}. \tag{3}$$

The $\omega_{\pm}$ are plotted as functions of $\omega_{\rm L} = r\omega_{\rm U}$ in Figure 6. When $r = \omega_{\rm L}/\omega_{\rm U}$ is large, ω_{-} approaches $\omega_{\rm U}$ and ω_{+} approaches $\omega_{\rm L}$. If $r \approx 1$ or $\omega_{\rm L}$ approaches $\omega_{\rm U}$, two curves of $\omega_{\pm}$ avoid crossing.

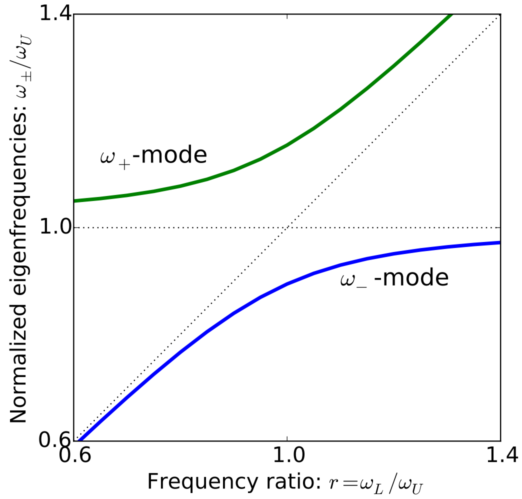


Figure 6. The two eigenfrequencies $\omega_\pm$ of the mechanical model are plotted as functions of $\omega_{\rm L} = r\omega_{\rm U}$, where $\alpha = 0.2$ is assumed. The dotted lines represent $\omega_{\rm U}$, $\omega_{\rm L}$. These cross each other at $r = 1$, whereas the flute resonance frequencies $\omega_\pm$ are changed as if they avoid crossing.

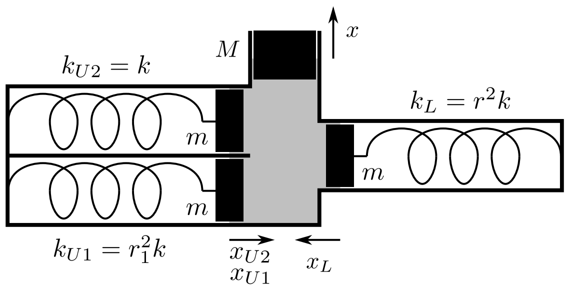


Figure 7. Mechanical model for explaining resonance frequencies of the recorder played with cross-fingering. The upper bore has two fixed resonance frequencies $\omega_{\rm U1}$ and $\omega_{\rm U2}$, where $r_1 = \omega_{\rm U1}/\omega_{\rm U2} \approx 0.5$ is a model parameter. The lower bore has a resonance frequency $\omega_{\rm L}$ varying with ratio $r = \omega_{\rm L}/\omega_{\rm U2}$.

4 MECHANICAL MODEL OF A RECORDER

The basic mechanism of pitch bending due to cross-fingering has been clarified so far. In this section, actual pitches at the second register on the alt recorder played with cross-fingering are explained. For this purpose, a mechanical model with three degrees of freedom as shown in Figure 7 is employed.

The upper bore is assumed to have only the lowest two resonance frequencies $\omega_{\rm U1}$ and $\omega_{\rm U2}$. These are almost harmonically related and a model parameter $r_1 \equiv \omega_{\rm U1}/\omega_{\rm U2} \approx 0.5$ is introduced. The lower bore has just one resonance frequency $\omega_{\rm L}$ as before. The ratio of this frequency to $\omega_{\rm U2}$ is written as $r = \omega_{\rm L}/\omega_{\rm U2}$. Displacements of the three mechanical oscillators are denoted by $x_{\rm U1}$, $x_{\rm U2}$ and $x_{\rm L}$. In the same way as in the last section, the eigenvalue equation of this mechanical model becomes

$$\begin{bmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{bmatrix}^{-1} \begin{bmatrix} r_1^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{bmatrix} \begin{bmatrix} x_{\rm U1} \\ x_{\rm U2} \\ x_{\rm L} \end{bmatrix} = \frac{\omega^2}{\omega_{\rm U2}^2} \begin{bmatrix} x_{\rm U1} \\ x_{\rm U2} \\ x_{\rm L} \end{bmatrix}. \tag{4}$$

By solving this equation, three eigenfrequencies can be obtained. These are plotted as functions of the lower

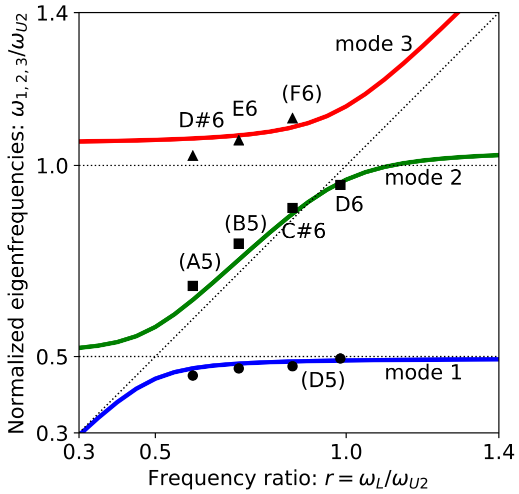


Figure 8. Three eigenfrequencies of the recorder's mechanical model as functions of the lower bore resonance frequency or the ratio $r = \omega_{\mathrm{L}}/\omega_{\mathrm{U2}}$. The model parameters are $\alpha = 0.25$ and $r_1 = 0.5$. Sound frequencies played on an actual alt recorder are also plotted with circle, square and triangle markers together with corresponding note names.

bore resonance frequency or $r = \omega_{\mathrm{L}}/\omega_{\mathrm{U2}}$ in Figure 8. The lower bore resonance frequency ω_{L} depicted by the dotted line inclined at 45 degrees intersects the two upper bore frequencies ω_{U1} and ω_{U2} shown with the horizontal dotted lines. In contrast, the eigenfrequencies of the recorder ω_1, ω_2 and ω_3 avoid crossing twice at $r = 0.5$, and 1. This is because the lower bore interacts with the upper bore twice near the first and second resonance frequencies.

An alt recorder was played with fingerings D6, C♯6, E6, and D♯6 in Table 1. The output sound was recorded and analyzed. The played frequencies are compared with the eigenfrequency curves in Figure 8. By changing the blowing pressure for each fingering, two or three pitches could be played. For example, if the recorder was played with normal D6 fingering and with normal blowing pressure, the pitch of D6 was gained as indicated by the square marker. This is identified as sounding in the second mode. If the blowing pressure was lowered, a pitch close to D5 was obtained as shown in the circle marker. This is sounding in the first mode. It was not possible to excite the third mode with fingering D6 even if the blowing pressure was increased. With cross-fingering of C♯6, the first and second resonance modes could be excited. The C♯6 sound results from the normal effect or pitch flattening due to cross-fingering. By just blowing harder, F6 in the third mode could not be played. However, F6 could be gained after playing E6 with fingering E6 by increasing the blowing pressure and at the same time by gradually changing the fingering to C♯6. The E6 sound that is one whole tone higher than the normal fingering was identified as the excitation in the third mode of the recorder. Furthermore, when fingering was changed to D♯6, the D♯6 sound that is a half tone higher was played. If the flute is blown softer in these fingerings, soundings in the first and second modes were also generated.

5 CONCLUSIONS

The resonance frequencies of the mechanical model and the frequencies actual played on the recorder were in good agreement. The usual pitch flattening and anomalous pitch sharpening due to cross-fingering on the recorder could thus be explained in a unified way with the model of three mechanical oscillators, two of which are associated with the upper bore resonances and one of which is with the lower bore resonance. The central mechanism is avoided crossing where the mechanical oscillators in the upper and lower bores are coupled by sharing the mass of the open tone hole.

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