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From the bifurcation diagrams to the ease of playing of reed musical instruments. A theoretical illustration of the Bouasse-Benade prescription? *†‡

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Abstract

Reed musical instruments can be described in terms of conceptually separate linear and nonlinear mechanisms: a localized nonlinear element (the valve effect due to the reed) excites a linear, passive acoustical multimode element (the musical instrument usually represented in the frequency domain by its input impedance). The linear element in turn influences the operation of the nonlinear element. The reed musical instruments are self-sustained oscillators. They generate an oscillating acoustical pressure (the note played) from a static over-pressure in the player's mouth (the blowing pressure).

A reed instrument having N acoustical modes can be described as a 2N dimensional autonomous nonlinear dynamical system. A reed-like instrument having two quasi-harmonic resonances, represented by a 4 dimensional dynamical system, is studied using the continuation and bifurcation software AUTO. Bifurcation diagrams are explored with respect to the blowing pressure, with focus on amplitude and frequency evolutions along the different solution branches. Some of the results are interpreted in terms of the ease of playing of the reed instrument. They can be interpreted as a theoretical illustration of the Bouasse-Benade prescription.

Keywords: Acoustics, reed instruments, bifurcation diagrams, resonance inharmonicity, ease of playing

1 INTRODUCTION

A quest for the grail of wind instruments musical acoustics is to try to understand keys of intonation and ease of playing. From the physic's modelisation point of view, we try to understand keys of what is controlling the playing frequency (for intonation) and the minimum mouth pressure to get oscillations (for ease of playing). A state of art can be found in the recent books dealing with acoustics of musical instruments: [Benade 1990], [Campbell and Greated 1987], [Fletcher and Rossing 1998], [Chaigne and Kergomard 2016]).

Often it has been pointed out how brass instruments generally have a flaring bore so designed that impedance peaks are well aligned in order to reach as close as possible an harmonic series. If this alignment is clearly important for intonation, it is important too to get stable periodic oscillations easy to play. The necessity of an alignment in an harmonic series is called here the 'Bouasse-Benade prescription' because of what Benade wrote in his famous book [Benade 1990], or in [Benade and Gans 1968]: "The usefulness of the harmonically related air column resonances in fostering stable oscillations sustained by a reed-valve was first pointed out by the french physicist Henri Bouasse in his book Instruments à vent". As a counter-example a horn was designed to provide an air column whose resonance frequencies (frequencies of maximum input impedance) were chosen to avoid all possible integer relations between them, this horn has been called 'tacet horn' in [Benade and Gans 1968]. This reed instrument has been made so that the conditions for oscillation would then be most unfavorable.

To try to illustrate the 'Bouasse-Benade prescription', we want to study the influence of the inharmonicity on the playing frequency and on the minimum mouth pressure to get oscillations. And to do so, we will use the

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bifurcation diagram representation, the control parameter being the mouthp ressure. The bifurcation diagrams are nice results to begin to answer this quest of the grail.

The wind instruments, and particularly the brass, exhibit many resonances, which makes very difficult to demonstrate the Bouasse-Benade prescription. A way to overtake that is to work on the easiest of the complicated resonators: a resonator having two quasi-harmonic resonance frequencies F_{res1} and F_{res2} where the deviation to harmonicity, the inharmonicity parameter Inh, is defined by $F_{res2} = 2F_{res1}(1+Inh)$. And to facilitate the theoretical investigations, we have chosen a low frequency model of inward striking reed instruments, the reed being assimilated to its stiffness only and being undamped. Note that if the resonances are exactly harmonic ($Inh = 0$) this theoretical problem can be analysed analytically, and two bifurcation diagrams have already been obtained (Figure 8 and 10 of [Dalmont et al 2000]).

In section 2 of this paper we present briefly the theoretical backgrounds, and particularly the equations of the elementary model of wind instruments. Then section 2 documents the procedure used to calculate bifurcation diagrams by using a continuation method, after having reformulated the two coupled equations of the model in a set of four coupled first order ODE equations. The influence of the inharmonicity on the bifurcation diagrams is shown and discussed in section 3. The bifurcation diagrams are analysed in the light of the threshold of periodic oscillations (to do a link with the ease of playing of the musicians, and with the Bouasse-Benade prescription), and the effect of the inharmonicity on the playing frequency is discussed too.

2 THEORETICAL BACKGROUNDS

2.1 Acoustical model

The model presented here and used in the present publication is labelled as elementary because a number of major simplifications are made in deriving it (see for example [Hirschberg et al 1995], [Fabre et al 2018]). The vibrating reeds or lips are modeled as a linear one degree of freedom oscillator. The upstream resonances of the player's windway are neglected, as is the nonlinear propagation of sound in the air column of the instrument. Wall vibrations are also ignored. Despite these simplifications, the elementary model is capable of reproducing many of the important aspects of performance by human players on realistic reed and brass instruments. The model is based on a set of three coupled equations, which have to be solved simultaneously to predict the nature of the sound radiated by the instrument. These three constituent equations of the model are presented hereafter.

Besides the control parameters defining the embouchure of the player, including the reed or lips parameters and the mouth pressure p_m, and the input impedance of the wind instrument, there are three variables in the set of the coming three equations as a function of the time t: the reed or lip opening height $h(t)$, the pressure in the mouthpiece of the instrument $p(t)$, and the entering volume flow into the instrument $u(t)$.

In order to describe the vibrating reeds or lips, the first of the three constituent equations of the elementary model is:

$$\frac{d^2h(t)}{dt^2} + \frac{\omega_r}{Q_r}\frac{dh(t)}{dt} + \omega_r^2(h(t) - h_o) = -\frac{p_m - p(t)}{\mu}. \tag{1}$$

In this equation, which describes the reeds or lips as a one degree of freedom (1DOF) mechanical oscillator, the symbols ω_r, Q_r, h_o and μ represent the angular reed resonance frequency, the quality factor of the reed resonance, the value of the reed or lip opening height at rest, and the effective mass per unit area of the reeed or lips respectively. These quantities are parameters of the model, which are either constant (in a stable note) or changing slowly in a prescribed way (in a music performance). Note that if μ is positive, an increase of the pressure difference $(p_m - p(t))$ will imply a closing of the reed or lips aperture. It is called the 'inward striking' model and is used mainly for reed instruments. If μ is negative, an increase of the pressure difference will imply an opening of the reed or lips aperture. It is called the 'outward striking' model and is used mainly for brass (lip reed) instruments.

The second constituent equation describes the relationship between pressure and flow in the reed lip channel:

$$u(t) = wh^{+}(t)\sqrt{\frac{2}{\rho}|p_m - p(t)|}sign(p_m - p(t)) \tag{2}$$

where the square root originates from the Bernoulli equation, and the positive part of the reed or lips aperture $h^{+} = max(h,0)$ models the closed reed or lips.

The third and last constituent equation describes the relationship between flow and pressure in the instrument mouthpiece. It is written in the frequency domain using the input impedance $Z(\omega)$ of the wind instrument:

$$p(\omega) = Z(\omega)u(\omega). \tag{3}$$

Other than the difference of sign of μ between 'inward striking' reed instruments model and 'outward striking' brass instruments model, there is another difference between these two subfamilies of wind instruments. The control parameter ω_r of vibrating lips is varying a lot, over four octaves, to get the entire tessiture of a given brass instrument. At the opposite the ω_r associated to reeds is more fixed (slightly varying because of the lower lip of the clarinet or saxophone player) and most of the time very large compared to the playing frequencies. This last fact justifies a low frequency approximation of the elementary model: the ω_r is assumed infinite and the reed undampted. In other words, the reed is reduced to its stiffness only and the set of three equations becomes a set of two equations as follows:

$$\begin{cases} u(t) = w[h_o - \frac{p_m - p(t)}{\mu\omega_r^2}]\sqrt{\frac{2}{\rho}(p_m - p(t))} \\ p(\omega) = Z(\omega)u(\omega). \end{cases} \tag{4}$$

When the mouth pressure is too high, the reed can be blocked against the lay of the mouthpiece. Then the 'closure pressure' defined by $p_M = \mu\omega_r^2.h_o$ is the minimal mouth pressure for which the reed remains closed in the static regime (h becomes equal to 0). By using this closure pressure, a dimensionless mouth pressure γ can be defined: $\gamma = p_m/p_M$ (note that most of the time the dimensionless mouth pressure is called abusively mouth pressure).

It is this elementary low frequency model for reed instruments which is used in the present paper. In the following, the nonlinear equation of the model is approximated by its third order Taylor series around the equilibrium position defined by $h_{eq} = h_o - \frac{p_m}{\mu\omega_r^2}$, $u_{eq} = wh_{eq}\sqrt{\frac{2}{\rho}p_m}$ and $p_{eq} = 0$:

$$u(t) = u_{eq} + A_1 p(t) + A_2 p(t)^2 + A_3 p(t)^3. \tag{5}$$

If we assume a non-beating reed which is typically obtained for a dimensionless mouth pressure γ lower than 0.5, the above third order approximation of the flow rates is appropriate.

The above elementary model based on the set of two equations has to be solved to predict the nature of the sound radiated by the instrument.

2.2 Continuation method

A nice way to have an overview of the dynamics over small and large amplitudes is to use the bifurcation diagram representation. A very few of them can be obtained analytically (see the previous subsection). It is possible to obtain bifurcation diagrams numerically for a large range of situations by using continuation methods, such as implemented in AUTO software [Doedel et al 1997] or MANLAB software [Karkar et al 2013]

for example. In order to use AUTO technique in the following section, the elementary model has to be mathematically reformulated in a set of coupled first order ODE equations.
To do so, we have mainly to introduce the derivative of the lips or reed position as a new variable and to reformulate the input impedance equation (Eqn. 3) by a sum of individual acoustical resonance modes in the frequency domain, and then to translate them in the time domain. There are two ways to manage that: sum of real modes (see for example [Debut et al 2004]), sum of complex modes (see for example [Silva et al 2014]). These two ways of approximating the input impedance in the frequency domain lead to two different sets of first order equations $\frac{dX}{dt} = F(X)$. In the present paper we use the 'real mode' representation of the input impedance Z.

The modal-fitted input impedance with N resonance modes, is written as follows:

$$Z(\omega) = \sum_{n=1}^{N} Z_n \frac{jq_n\omega\omega_n}{\omega_n^2 + jq_n\omega\omega_n - \omega^2}. \tag{6}$$

where the n^{th} resonance is defined by three real constants, the amplitude Z_n, the dimensionless damping coefficient q_n and the angular frequency ω_n.
Each term of Eqn. 6 can be written in the time domain as follows:

$$\frac{d^2 p_n}{dt^2} + q_n\omega_n \frac{dp_n}{dt} + \omega_n^2 p_n(t) = Z_n q_n \omega_n \frac{du}{dt}, \tag{7}$$

such that the acoustical pressure is $p(t) = \sum_{n=1}^{N} p_n(t)$.
Taking into account the other equation of the elementary model, the derivative of the volume flow nonlinear equation (Eqn. 4), the previous set of N second order ODE (Eqn. 7) can be rewritten by using the following expression of $\frac{du}{dt}$:

$$\begin{aligned}\frac{du}{dt} &= w\frac{1}{\mu\omega_r^2}\frac{d(\sum_{n=1}^{N} p_n)}{dt}\sqrt{\frac{2}{\rho}(p_m - \sum_{n=1}^{N} p_n(t))} \\ &+ w[h_o - \frac{p_m - \sum_{n=1}^{N} p_n(t)}{\mu\omega_r^2}](-1/2)\frac{d(\sum_{n=1}^{N} p_n)}{dt}\frac{1}{\sqrt{\frac{2}{\rho}(p_m - \sum_{n=1}^{N} p_n(t))}}.\end{aligned} \tag{8}$$

If the third order Taylor series approximation is used (Eqn. 5), then in place of Eqn. 8 we get:

$$\frac{du}{dt} = [A_1 + 2A_2(\sum_{n=1}^{N} p_n) + 3A_3(\sum_{n=1}^{N} p_n)^2]\frac{d(\sum_{n=1}^{N} p_n)}{dt}.$$

Then the equations can be put into a state-space representation $\frac{dX}{dt} = F(X)$, where F is a nonlinear vector function, and X the state vector having $2N$ real components defined as follows:

$$X = \left[p_1;\ldots;p_N;\frac{dp_1}{dt};\ldots;\frac{dp_N}{dt}\right]'. \tag{9}$$

In practice, because our paper is dedicated to a two quasi harmonically resonances instrument, the state space representation is based on the state vector of 4 real components $X = \left[p_1;p_2;\frac{dp_1}{dt};\frac{dp_2}{dt}\right]$.
Bifurcation diagrams for different values of inharmonicity are presented and discussed in the coming Section. It is assumed the following configuration of relative amplitudes between Z_1 and Z_2 of the two resonances: Z_1 slightly higher than Z_2.

3 BIFURCATION DIAGRAMS

3.1 Exactly harmonic resonances

The results shown in Figure 1 for the case $Inh = 0$ are qualitatively consistent with the one published in [Dalmont et al 2000] (see in particular its Figure 8). Note that the continuation method gives an additional information: the stability nature of the periodic oscillations. The bifurcation diagram shows two branches coming from the equilibrium position:

- the first branch is originating from the linear threshold $p_m = p_{thr1}$, associated to the first resonance F_{res1}, according to an inverse Hopf bifurcation. This fundamental regime is a 'standard Helmholtz motion' according to [Dalmont et al 2000]. Because it is an inverse bifurcation case, the branch is unstable and then becomes stable at the limit point at $p_m = p_{subthr}$.
- the second branch is originating from the linear threshold $p_m = p_{thr2}$, associated to the second resonance F_{res2}, according to a direct Hopf bifurcation. Note that p_{thr2} is bigger than p_{thr1}, because Z_1 is larger than Z_2. This branch which would correspond to the second regime (the octave) is not observable in practice, because the periodic solutions are unstable.

There is a third branch wich is originating from the unstable octave branch, thanks to a period doubling bifurcation. This branch which would correspond to an other fundamental regime (the 'inverted Helmholtz motion' according to [Dalmont et al 2000]) is unstable. The associated lower curve shows the frequency of the periodic oscillations corresponding to the stable branche of the bifurcation diagram. In particular the frequency of the fundamental regime (green curve) is locked at the value $F_{res1} = F_{res2}/2$ whatever p_m is.

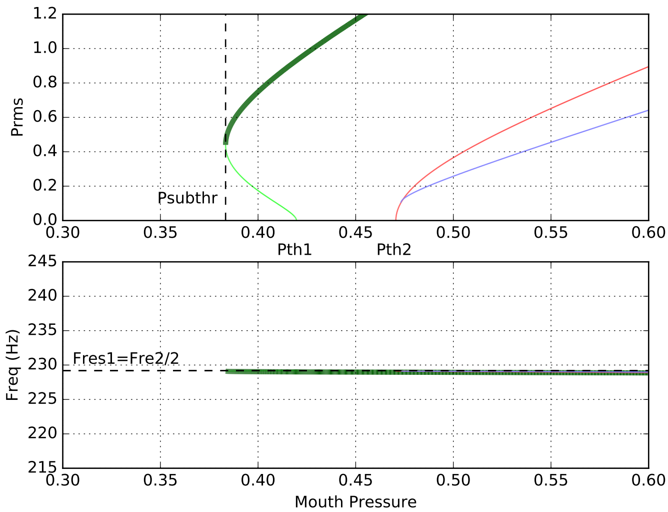

Figure 1. Bifurcation diagram for an air column with two exactly harmonic resonances ($Inh = 0$) as a function of the control parameter p_m. Thick lines describe stable solutions, thin lines unstable solutions. Upper graph: RMS value of oscillating pressure $p(t)$. Lower graph: frequency of oscillation.

3.2 Quasi-harmonic resonances

Let's go to the case $Inh = 0.02$ (Figure 2). The bifurcation diagram is quite close to the one with $Inh = 0$. We would like to point out two things. First, at the threshold $p_m = p_{thr1}$ the Hopf bifurcation has become direct as it can be predicted theoretically ([Grand et al 1997]). Second, again there are periodic oscillations for mouth pressures p_m values below $p_m = p_{thr1}$ (though the bifurcation is direct) until a new value $p_m = p_{subthr}$ which is a bit larger than the one of the case $Inh = 0$. Note that the frequency of the fundamental regime (green curve) is not locked at the value F_{res1} anymore but is partially pull out toward the value $F_{res2}/2$, which is intuitively sensible. If the inharmonicity was negative, we would have the same kind of results, the frequency being pull out toward $F_{res2}/2$ lower than F_{res1}.

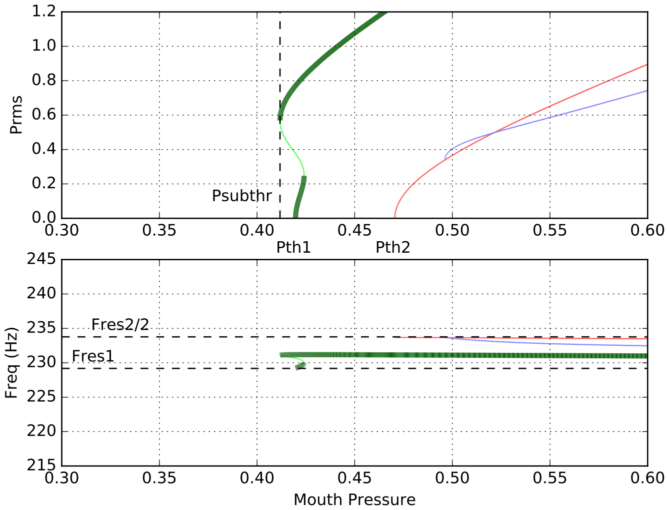

Figure 2. Bifurcation diagram for an air column with two quasi-harmonic resonances ($Inh = 0.02$) as a function of the control parameter p_m. Thick lines describe stable solutions, thin lines unstable solutions. Upper graph: RMS value of oscillating pressure $p(t)$. Lower graph: frequency of oscillation.

Let's go to the case $Inh = 0.04$ (Figure 3). Now the branch coming from the threshold $p_m = p_{thr1}$, corresponding to the fundamental regime, looks like a classical branch associated to the direct Hopf bifurcation, there is no p_{subthr} anymore. $p_m = p_{thr1}$ is now the the threshold of oscillation. In fact, when the inharmonicity is increasing, the dynamics of the system behaves more and more like the dynamics of the one acoustical resonance system. The frequency of the fundamental regime comes from the threshold value $F_{thr1} = F_{res1}$ at the direct Hopf bifurcation point, and then is partially pull out toward the value $F_{res2}/2$.

The above discussion illustrates important things because of the inverse Hopf bifurcation:
- on one hand, there is a minimum value $p_m = p_{subthr}$ lower than p_{thr1} where we can have periodic oscillations. This particular value p_{subthr} is a kind of quantitative caracterisation of the ease of playing. It has been shown that the lowest value of p_{subthr} is obtained when the two resonances are perfectly harmonic ($Inh = 0$). It suggests the reed instrument is the easiest to play when $Inh = 0$. In a way this is a theoretical illustration of the Bouasse-Benade prescription ([Bouasse 1929], [Benade 1990]).
- on the other hand, the stable periodic oscillations which appear for p_m slightly larger than p_{subthr} can have fundamental frequencies significantly different from $F_{thr1} = F_{res1}$ because of the effect of the presence of the second resonance which controls partially the intonation of the fundamental regime. This study illustrates the limit of the linear stability analysis approach to predict the behavior of the small amplitude periodic oscillations. Note that it is the case too when $Z_2 > Z_1$ (see Figure 10 of [Dalmont et al 2000]).

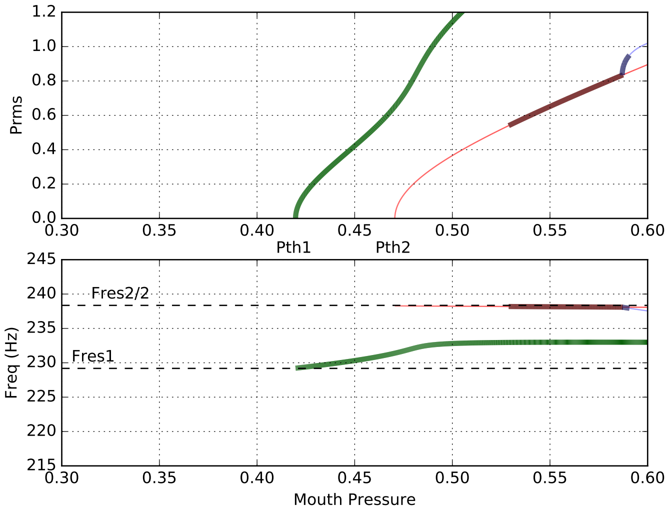

Figure 3. Bifurcation diagram for an air column with two quasi-harmonic resonances ($Inh = 0.04$) as a function of the control parameter p_m. Thick lines describe stable solutions, thin lines unstable solutions. Upper graph: RMS value of oscillating pressure $p(t)$. Lower graph: frequency of oscillation.

4 CONCLUSIONS AND PERSPECTIVES

Bifurcation diagrams of a basic reed instrument having two quasi-harmonic resonances have been calculated by using a continuation method. The dynamical behaviour has been described as a function of the inharmonicity between the two acoustical resonancies, from perfect harmonicity to an inharmonicity equal to 0.04. The Hopf bifurcations, direct or inverse, are observed, the stability of the branches analysed, and their implication on the playing frequency discussed. Some of the mouth pressure thresholds results are interpreted in terms of the ease of playing of the reed instrument. Because of an inverse Hopf bifurcation (perfect harmonicity) or of a double fold after a direct Hopf bifurcation (moderate inharmonicity), there may be a minimum value $p_m = p_{subthr}$ lower than p_{thr1} where periodic stable oscillations can be observed. This particular value p_{subthr} is a kind of quantitative caracterisation of the ease of playing. It has been shown that the lowest value of p_{subthr} is obtained when the two resonances are perfectly harmonic. It suggests the reed instrument is the easiest to play when the resonances are harmonic. In a way this is a theoretical illustration of the Bouasse-Benade prescription ([Bouasse 1929], [Benade 1990]).

Obviously the results given in the present manuscript are depending on a physical model of reed or brass instruments based on approximations which sometimes can be perceived as basic or even crude. For example the non linear equation describing the entering volume fow has been approximated by its third order Taylor series expansion. Preliminary bifurcations diagrams have been successfully obtained by using the exact non linear equation too. The present study which is a preliminary one, offers many possibilities of complementary works which are in progress: same study with the assumption Z_1 slightly lower than Z_2, dynamics of the reed taken into account, more than two resonances in order to study the brass instruments.

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