Investigation of the resonance frequency of a cavitating vortex

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Introduction

The cavitation pattern of a modern ship propeller is often characterized by a cavitating vortex. This cavitating vortex generates low frequency hull pressure fluctuations which may cause noise and vibration hindrance on board the ship. Analysis of the signals suggests the presence of specific frequency components [1], but detailed investigation is hindered by the complexity of the problem. On the theoretical side, a dispersion relation for waves on cavitating vortices has been developed [2,3] but the applicability of this relation is still unknown. The solution in the limit of small axial wave numbers has been used with some success for comparison with resonance frequencies found in experimental data [4,5] but a motivation for the approach is missing. The paper proposes an alternative procedure in which the method of stationary phase is used to predict resonance frequencies. The approach is validated using measurement data by Maines & Arndt [4] who found resonance frequencies for a cavitating vortex generated by a foil in a cavitation tunnel.

Derivation dispersion relation

For the theoretical analysis of resonance frequencies use can be made of the dispersion relation for an inviscid cavitating vortex as originally derived by Lord Kelvin [2]. The formulation for incompressible flow of Lord Kelvin has been extended to compressible flow for acoustic analysis by Morozov [3]. A further extension with a constant axial free-stream velocity component is presented by Bosschers [6], which includes a discussion on radiated noise aspects. A small review of the theory is presented here, followed by a new extension to include viscous effects.

The starting point for the derivation of the dispersion relation is the convected Helmholtz equation for a disturbance velocity potential $\tilde{\varphi}$ [7],

$$\nabla^2\tilde{\varphi}-\frac{1}{c^2}\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\nabla\right)^2\tilde{\varphi}=0, \qquad [1/s] \qquad (1)$$

in which c denotes the speed of sound and $\mathbf{U}$ the free-stream velocity vector. The free-stream flow needs to be irrotational with a velocity magnitude much smaller than the speed of sound. A cylindrical coordinate system (r,θ,z) is adopted with a harmonic variation of the disturbance potential given by

$$\tilde{\varphi}=\phi(r)\,e^{i(k_z z+n\theta-\omega t)}, \qquad [m^2/s] \qquad (2)$$

in which k_z corresponds to the flexural wave number in axial direction, n the azimuthal wave number which must be an integer and ω the angular frequency. If only the free-stream axial velocity component W is considered, the solution for the disturbance potential is given by the Hankel function of the first kind:

$$\tilde{\varphi}=\hat{\phi}\,H_n^1(k_r r)\,e^{i(k_z z+n\theta-\omega t)}, \qquad [m^2/s] \qquad (3)$$

where k_r corresponds to the projected acoustic wave number in radial direction:

$$k_r^2=\frac{1}{c^2}(\omega-Wk_z)^2-k_z^2 \quad . \qquad [1/m^2] \qquad (4)$$

The addition of an azimuthal velocity will lead to a Matthieu equation for which no analytical solution is available. However, the additional terms are small for modes n= 1 and n= 2 for low frequencies and can be neglected. For the mode n= 0 no additional terms arise and the solution is given by equation (3).

The distortion of the cavitating vortex with average radius r_c is described by a number of modes characterized by again k_z, n, ω and an amplitude $\hat{r}$. The local cavity radius η is given by

$$\eta=r_c+\hat{r}\,e^{i(k_z z+n\theta-\omega t)}. \qquad [m] \qquad (5)$$

The mode $n=0$ corresponds to a breathing mode and involves volume variations. Mode $n=1$ corresponds to a serpentine mode, also called bending mode, helical mode or displacement mode as it is the only mode which leads to a displacement of the vortex centre line. The mode $n=2$ is the bell mode or double helix or fluted mode and leads to an elliptical shape of the vortex core. A visualization of the modes is given in Figure 1. The distortions are transverse propagating inertial waves and are often referred to as Kelvin waves [8].

For the analysis of the boundary conditions, the unperturbed velocities in the cylindrical coordinate system are given by:

$$\mathbf{U}=(U,V,W)=\left(0,\frac{\Gamma}{2\pi r},W\right), \qquad [m/s] \qquad (6)$$

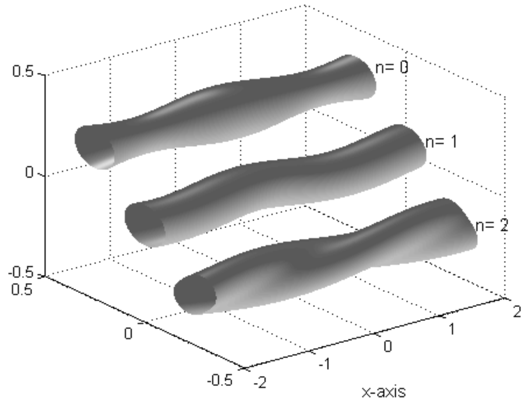

Figure 1: Visualisation of the spatial deformation of a cavitating vortex for the modes n=0, 1 and 2 at one time-step.

in which the azimuthal velocity component due to a 2-D vortex filament is used.

The boundary conditions are given by the kinematic boundary condition which states that the deformed cavity surface is a stream surface and by the dynamic boundary condition which states that the pressure at the cavity surface equals vapour pressure (neglecting surface tension). Combination of the linearized kinematic and dynamic boundary condition with the solution of the convected Helmholtz equation leads to the following dispersion relation:

$$\omega_{1,2} = Wk_z + \Omega\left[n \pm \sqrt{\frac{-k_r r_c H_n^{1\prime}(k_r r_c)}{H_n^1(k_r r_c)}}\right], \qquad \text{[rad/s]} \quad (7)$$

where the azimuthal velocity at the cavity V_c is replaced by the vortex angular frequency Ω defined as

$$\Omega = \frac{\Gamma}{2\pi r_c}\frac{1}{r_c} = \frac{V_c}{r_c}. \qquad \text{[rad/s]} \quad (8)$$

The numerator in the square root in equation (7) contains the derivative of the Hankel function. Each mode contains two frequencies corresponding to the plus and minus sign in the right-hand-side. In the following, this sign is also used to identify the mode.

The criterion for a sound wave to occur is that the radial wave number squared, defined by equation (4), is larger than zero. If it is smaller than zero, the radial wave number becomes an imaginary number. An evanescent wave in radial direction now occurs and the Hankel function reduces to the MacDonald function K. This is the situation for the range of relative low frequencies in the current application area. Sound is then radiated due to the finite length of the vibrating cavity, as shown by the analytical formulation for the far field radiated noise due to standing waves of small amplitude on a cylinder of finite length as presented by Junger & Feit [9]. The formulation also shows that the maximum pressure amplitudes of different modes n scale as $(k\,r_c)^n$, with k the acoustic wave number. For the low frequencies and small cavity diameters considered, always $(k\,r_c) << 1$, hence the higher order modes are not expected to contribute to the radiated noise and only the breathing mode $n = 0$ is expected to be important.

The potential flow outside the cavitating vortex allows the application of the Bernoulli equation and the azimuthal velocity at the cavity can be related to the difference between the free-stream pressure p_∞ and vapour pressure p_v. Upon introducing the cavitation number σ_V, the following formulation is found for the mean non-dimensional azimuthal velocity at the cavity radius

$$\frac{\Omega r_c}{W} = \frac{V_c}{W} = \sqrt{\frac{p_\infty - p_v}{0.5\rho W^2}} = \sqrt{\sigma_V}. \qquad \text{[-]} \quad (9)$$

The non-dimensional form of the dispersion relation for low frequencies is then given by

$$\bar{\omega}_{1,2} = \frac{\omega_{1,2} r_c}{W} = \kappa + \sqrt{\sigma_V}\left[n \pm \sqrt{\frac{-\kappa K_n'(\kappa)}{K_n(\kappa)}}\right], \qquad \text{[-]} \quad (10)$$

in which a non-dimensional wave number $\kappa = k_z r_c \cong i\,k_r r_c$ has been introduced as for low frequencies and small axial velocities the magnitude of the radial wave number becomes approximately equal to the magnitude of the axial wave number. The dispersion relation gives only real values for the frequency and the distortions are therefore neutrally stable.

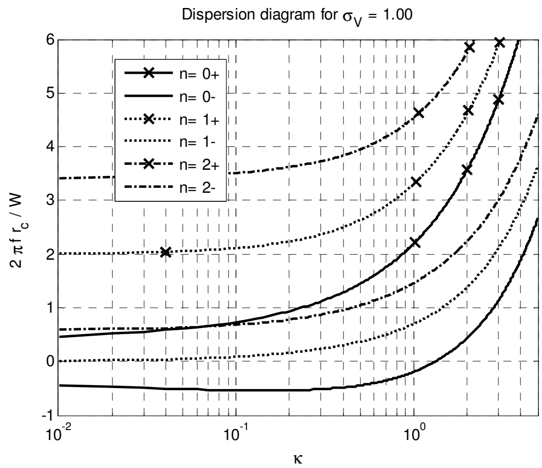

Figure 2: The dispersion diagram for a cavitating vortex

The variation of the non-dimensional frequency $\bar{\omega}$ with non-dimensional axial wave number at prescribed cavitation number is plotted in Figure 2. It is seen that the frequency variation at all modes becomes small for small wave numbers. Except for both breathing modes, the group speed becomes constant at small wave numbers, indicating asymptotic convergence to a constant frequency for decreasing wave number. The frequency of the two breathing modes becomes zero for vanishing wave number. The negative frequency observed for mode $n = 0^-$ indicates that the distortions are propagating in opposite direction to the free-stream velocity. The distortions for mode $n=1^-$ always propagate at a speed close to the free-stream flow velocity. As the group speed also has a value close to the free-stream velocity, it is concluded that dispersion effects are very small for this mode. At smaller wave numbers, the frequency becomes zero and deformations for this mode become stationary and can easily be observed during cavitation tests. It is seen that there is a coinciding frequency for the modes $n=0^+$ and $n=2^-$ which may cause an interaction between them. Most interesting however, is that there is a wave number of the mode $n=0^-$ for which the frequency has a local minimum which is similar to zero group speed. This condition corresponds to the dominant wave number in an asymptotic analysis for large time using the method of stationary phase [10] and it shows the existence of a dominant frequency component. The corresponding frequency is negative indicating that the distortions are propagating upstream. For larger cavitation numbers the group speeds of the modes $n=1^-$ and $n=2^-$ may become zero as well.

Correction for viscous effects

The resonance frequency using equation (10) with the zero group speed condition is valid for potential flow only. Comparison of this frequency with experimental data already gives a reasonable correlation [6] but it was expected that further improvement could be obtained by inclusion of viscous effects. Instead of performing a full 3D stability analysis for viscous flow a simple correction method is used for the azimuthal velocity at the cavity radius given in equations (8) and (9). An analytical formulation for a 2D axisymmetric cavitating vortex in viscous flow has been developed [11] which is an extension of the formulation of a (non-cavitating) Lamb-Oseen vortex. The azimuthal velocity at radial location r is given by

$$v(r)=\frac{\Gamma}{2\pi r}\left\{1-\frac{r_v^2}{r_v^2+\zeta r_c^2}\exp\left[-\zeta\frac{\left(r^2-r_c^2\right)}{r_v^2}\right]\right\}, \qquad \text{[m/s]} \quad (11)$$

where r_v corresponds to the radius of the viscous core, r_c to the radius of cavitating core and ζ is a constant such that the maximum azimuthal velocity for a non-cavitating vortex occurs on the radius of the viscous core, $\zeta=1.2564$. The equation can be used to derive an analytical expression for the pressure which is a function of the same parameters and which is given in [11]. The azimuthal velocity at the cavity radius now corresponds to

$$v\left(r_c\right)=V_c=\frac{\Gamma}{2\pi r_c}\left\{\frac{\zeta r_c^2}{r_v^2+\zeta r_c^2}\right\}, \qquad \text{[m/s]} \quad (12)$$

which is used instead of equation (9) with a procedure discussed later. The viscous effects always lead to a reduction of the azimuthal velocity. The equations for the azimuthal velocity and pressure can be combined such that the non-dimensional azimuthal velocity becomes a function of cavitation number and the ratio of the parameters r_v and r_c. It is expected that this simple correction procedure for the resonance frequency is only valid for $r_c/r_v>1$.

Comparison with experiment

Resonance frequencies in the radiated noise of cavitating vortices generated by wings in cavitation tunnels have been reported by Maines & Arndt [4] and Briançon-Marjollet & Merle [5]. Both found reasonable correlation with the presented dispersion relation by using the original formulations of Lord Kelvin and Morozov in the limit $\kappa<<1$, but the best correlation was obtained for higher order modes $n>0$ which is somewhat surprising considering their low radiation efficiency. Here, the experimental results of Maines & Arndt are used as their results are described in greatest detail. The test-case was first reported by Higuchi et al. [12] and has been further investigated by Maines & Arndt. Different hydrofoils of elliptical planform were tested in two different cavitation tunnels, one in Obernach, Germany and one at St. Anthony Falls Laboratory (SAFHL), Minneapolis, USA. In both tunnels sound was radiated at a very distinct frequency component for a small range of cavitation numbers which could be related to the cavitating vortex through high speed video. This 'singing' only occurred when the cavitating vortex was attached to the blade surface. The frequency was found to vary between 400 Hz and 1.1 kHz, depending on tunnel velocity and lift coefficient. Two different waves were observed on the cavitating vortex, one wave showing stationary nodes while the other wave oscillates in the axial direction. A photograph of the cavitating vortex is given in Figure 3.

Figure 3: Observation of the singing vortex [4].

The viscous correction has been applied by selecting the vortex circulation such that the cavity diameter equals 4 mm for $\sigma_V=1$ which are the mean values of the experimental data. The viscous core size was obtained by scaling data of [13] to the Reynolds number of the Obernach measurements using the formulation for a flat plate turbulent boundary layer. This results in a viscous core diameter of 1.8 mm. The cavity diameter was then varied between 2.5 and 7 mm and the corresponding relation between azimuthal velocity at the cavity radius and cavitation number was used instead of Equation (9).

The variation of the non-dimensional frequency with cavitation number is presented in Figure 4. It is seen that the experimental data falls between the zero group speed curves for potential flow and with viscous correction. Figure 4 also shows the frequency at which the modes $n = 0^+$ and $n = 2^-$ coincide, but the correlation with experimental data is less than for the zero group speed condition. In addition, the non-dimensional wave numbers of the zero group speed shows much better correlation with the experimental data than for the coinciding frequency and wave number. This data is not presented here.

Figure 4 suggests that the singing is caused by the mode n=0- which corresponds to the point of stationary phase. Because of the negative frequency, see Figure 2, the distortions are travelling upstream from the vortex towards the sheet on the foil which then starts to grow and shrink again. Without collapse of either sheet or vortex cavity, the continuous interaction between the sheet and vortex cavity cause the continuous singing.

The double helical structure might be due to the elliptical shape of the cavity on the blade which is shed downstream to form an elliptical vortex structure, known as a Kirchhoff vortex [8]. This elliptical shape might then also cause the appearance of stationary nodes when viewing the vortex from one side only. Comparison of the axial wavelengths shows however large differences between theory and reported experimental data in [4], so this aspect requires further study. It is finally remarked that the theoretical vortex model is a very simple representation of the actual vortex and it is thus only allowed to analyse the overall trends of the data.

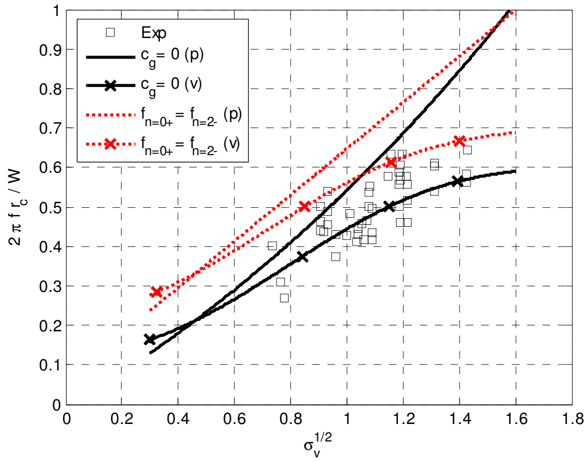

Figure 4: Comparison of the non-dimensional resonance frequency between the experimental data of Maines & Arndt [4], symbols, and theoretical curves for potential flow (p) and potential flow corrected for viscous effects (v). The theoretical curves are for zero group speed $(c_g = 0)$ and coinciding frequency.

Concluding remarks

A new formulation for the occurrence of resonance frequencies of cavitating vortices has been derived. It is based on the stationary phase solution given by the zero group speed condition of the dispersion relation for cavitating vortices in potential flow. The solution is the breathing mode $n = 0^-$ which is an efficient noise source. Good agreement with experimental data is obtained if a simple correction for viscous flow is applied. The relevance of other aspects described by the dispersion relation needs to be further investigated.

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