A Semi-analytical model for rooms with absorptive boundary conditions

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Introduction

For the prediction of sound fields in acoustic volumes typically energy methods, like the Statistical Energy Analysis [2] are used. The SEA is a very robust method for subsystems with high modal density, if the averaging over frequency bands, points of excitation and points of observation is carried out. Due to the loss of phase information its performance is limited if a higher resolution of the spatial distribution of the sound field is of interest or if the influence of boundary conditions should be studied. It even fails when Impulse Response Functions (IRFs), which are used for classification and optimization of acoustic cavities (e.g. rooms, designed for speech or music), shall be calculated. Optimization is done by placing acoustic elements like reflectors or absorbers (passive absorbers or plate resonators) into the room.

An efficient way, especially for the optimization of the position of these acoustic elements, is to calculate the IRFs of the coupled system (fluid-structure) in the frequency domain applying a Component Mode Synthesis. In this paper a method is presented in order to calculate the steady state response of an acoustic volume with a compound absorber as a boundary condition, which is loaded by a harmonically oscillating pressure load.

System and Method

The method presented in this paper is described on the basis of a rectangular room, loaded by a harmonically oscillating pressure load. Its dimensions are $[0, L_x], [0, L_y], [0, L_z]$ and its walls are reflective with the exception of the boundary at $x = 0$. A compound absorber, consisting of a porous foam between two layers of homogeneous material is mounted there (Figure 1).

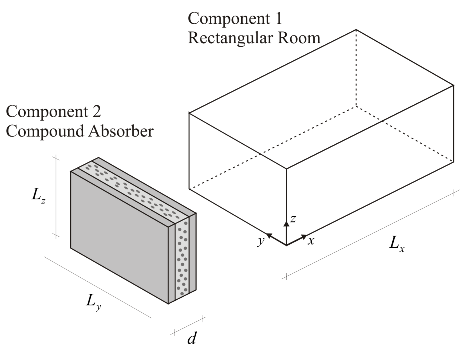


Figure 1: System and individual components

In the following the steady state response of the coupled problem under a harmonically oscillating pressure load, which is given as

$$p^{Load}(x,y,z,t) = \frac{1}{2} p(x,y,z) \left(e^{(i\Omega t)} + e^{(-i\Omega t)} \right)$$

is derived. A Ritz Method is applied for the coupled system, where approaches are used for the velocity of the acoustic fluid (component 1), which fulfill the boundary conditions at the reflective walls and provide the interaction with the absorptive structure (component 2).

Acoustic Fluid

The acoustic fluid (air) in a room follows the wave equation:

$$\triangle p_L - \frac{1}{c_L^2} \frac{\partial^2 p_L}{\partial t^2} = 0 \quad (1)$$

The subscript $(\ldots)_L$ specifies the air. c_L is the speed of sound.

In the scope of a Ritz Method a product-approach is used for the spatial and the time dependent state variables pressure p_L and velocity $\mathbf{v}_L$. The acoustic-field velocity potential Φ_L is introduced.

$$\Phi_L(x,y,z,t) = \phi_L^+(x,y,z)\, e^{(i\Omega t)} + \phi_L^-(x,y,z)\, e^{(-i\Omega t)}$$

ϕ^+ and ϕ^- are conjugate complex numbers. For the given problem, the function for the spatial description ϕ_L^+ is chosen as:

$$\begin{aligned} \phi_L^+(x,y,z) &= \sum_{l=0}^{l_{max}} \sum_{m=0}^{m_{max}} \sum_{n=0}^{n_{max}} A_{lmn} \Theta_{l,m,n}^N(x,y,z) + \\ &+ \sum_{m=0}^{m_{max}} \sum_{n=0}^{n_{max}} B_{mn} \Theta_{m,n}^C(x,y,z) \end{aligned} \quad (2)$$

with:

$$\Theta_{l,m,n}^N(x,y,z) = \cos\left(\frac{l\,\pi}{L_x} x\right) \cos\left(\frac{m\,\pi}{L_y} y\right) \cos\left(\frac{n\,\pi}{L_z} z\right) \quad (3)$$

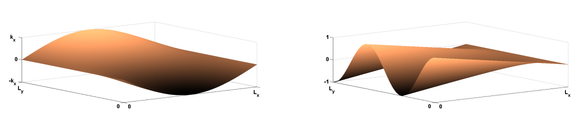
$$\Theta_{m,n}^C(x,y,z) = \left(x - \frac{x^2}{2\,L_x}\right) \cos\left(\frac{m\,\pi}{L_y} y\right) \cos\left(\frac{n\,\pi}{L_z} z\right) \quad (4)$$

The individual terms of equations (2) - (4) are derived in [3]. The velocity $\mathbf{v}_L$ can be calculated with the help of Φ_L (equation (2)) and the pressure p_L is derived from the velocity field $\mathbf{v}_L$:

$$\mathbf{v}_L = \operatorname{grad} \Phi_L \qquad (5)$$

$$\frac{\partial p_L}{\partial t} = -\rho_L\, c_L^2 \operatorname{div}(\mathbf{v}_L) \qquad (6)$$

Due to the reflective boundary conditions the mode-shapes are fixed interface modes (equation (3)). They are called normal modes in the CMS Method (superscript $(\ldots)^N$). Because of the impedance boundary condition (a compound absorber, mounted at a wall), also constraint modes (superscript $(\ldots)^C$) are introduced. They are defined for the subsystem of the rectangular room and fulfill the reflective boundary conditions at all surfaces of the room, with the exception of the interface at $x = 0$, which is considered as a free interface for their derivation [3]. Figure 2 shows exemplarily one normal mode and one constraint mode for the fluid velocity v_x in x-direction. They fulfill the boundary conditions at the reflective walls.



Normal mode Constraint mode

Figure 2: The normal mode ($l = 1, m = 1, n = 0$) and the constraint mode ($m = 2, n = 0$) for the fluid velocity v_x in x-direction of the rectangular room for an arbitrary section in z-direction and $n = 0$

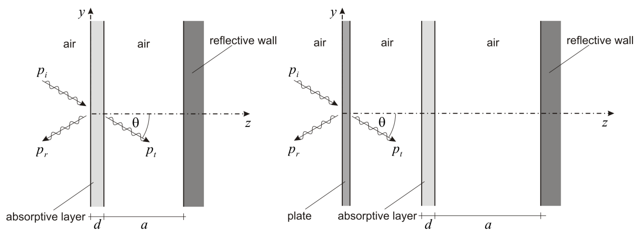
In the scope of this paper the compound absorber covers one wall completely. A coupling with a part of a wall is possible, but this has to be considered in the choice of the constraint modes as well as in the approach for the surface velocity of the compound absorber, which is derived later.

Compound Absorber

Absorbers used in room acoustics can be classified in two types, passive absorbers and plate- or Helmholtz-resonators (Figure 3). The technique, presented in this paper, permits an implementation of passive absorbers, consisting of porous materials like foams, mineral wool or cellular glass, and plate resonators. In order to describe these kinds of absorptive structures, a layered model, where homogeneous material, porous material and air are combined, is used. This model was presented at DAGA 2008 [5] and is described in detail in [6]. Thus the proceeding is only sketched briefly.

The compound absorber is defined in its local coordinate system (Figure 3). For assembling the full model, the impedance, describing component 2, (Figure 1) which is derived in the following section, will be transformed into the global coordinate system.

The plate structure consisting of layers of homogeneous, linear-elastic, isotropic material is described by the Lamé



Passive absorber Plate resonator

Figure 3: Classification of absorbers

Equation.

$$\left(\lambda^H + \mu^H\right) \operatorname{grad} \operatorname{div} \mathbf{u}_H + \mu^H\, \Delta \mathbf{u}_H = \rho^H\, \frac{\partial^2 \mathbf{u}_H}{\partial t^2} \qquad (7)$$

The superscript $(\ldots)^H$ denotes the homogeneous material in order to distinguish from the constituents of the porous medium. For modeling porous layers of an absorber, the linear Theory of Porous Media is applied. The porous medium, consisting of the constituents solid and gas (air), is modeled as a smeared volume with statistically distributed pores, whereas the solid phase spans the control space [1]. The porous solid is assumed to be much stiffer than the air. This results in a structural compressibility of the material, which consists of an incompressible porous solid and a compressible gas. The porous material, used in the absorber, obeys the following system of differential equations [1, 6]:

$$\begin{aligned} -n^S\, \nabla p + \left(\tilde{\lambda}^S + \mu^S\right) \operatorname{grad} \operatorname{div} \mathbf{u}_S + \mu^S\, \Delta \mathbf{u}_S + \\ + S_G\left(\mathbf{v}_G - \mathbf{v}_S\right) = \rho^S \mathbf{a}_S \end{aligned} \qquad (8)$$

$$-n^G\, \nabla p - S_G\left(\mathbf{v}_G - \mathbf{v}_S\right) = \rho^G \mathbf{a}_G$$

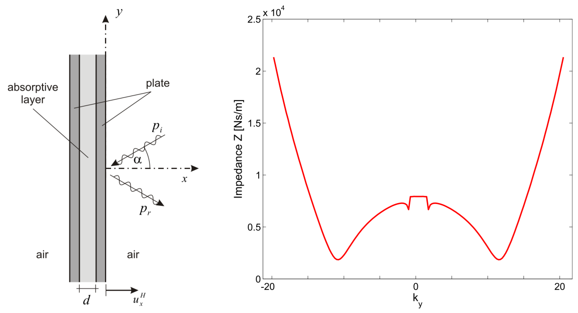
$$\frac{n^G}{R\,\theta}\, \frac{\partial p}{\partial t} + \rho_0^{GR}\, n^G \operatorname{div}(\mathbf{v}_G) + \rho_0^{GR}\, n^S \operatorname{div}(\mathbf{v}_S) = 0$$

The superscripts $(\ldots)^S$ and $(\ldots)^G$ denote parameters of the solid- and the gas- phase. n^α are the volume fractions, $\tilde{\lambda}^S$ and μ^S are the macroscopic Lamé constants. The superscript $(\ldots)^R$ specifies the real parameters to distinguish from parameters referred to the mixture. The homogeneous and porous layers, discussed in the previous sections, interact within the model of the porous absorber with the acoustic fluid, which is described by the wave equation (equation (1)).

The equations, specified above for the homogeneous and the porous material, are simplified expressing the (continuous differentiable) displacement field with a scalar potential Φ and a vector potential $\mathbf{\Psi}$ [6].
The partial differential equations (PDEs) as well as the systems of PDEs can be transformed into ordinary differential equations (ODEs) and systems of ODEs respectively, examining the problem in the wavenumber-frequency domain using Fourier Transformation methods. For a harmonic excitation the time t is transformed into the frequency domain ($t \circ\!\!-\!\!\bullet\, \Omega$). The spatial coordinates x and y are transformed into the wavenumber domain ($x \circ\!\!-\!\!\bullet\, k_x$, $y \circ\!\!-\!\!\bullet\, k_y$), assuming infinite dimensions in two directions. The disadvantage

of this transformation is the loss of the possibility to vary the system parameters in x- and y-direction. However for most applications in room acoustics a definition of different layers in the z-direction is sufficient.
The corresponding equations (1), (7) and (8) finally result in linear ODEs of second order with constant coefficients and can be solved by an exponential approach $e^{\lambda z}$. The unknown coefficients of the fundamental systems are determined using the boundary conditions between the different layers of the absorber [6].

The fluid-structure-interaction of the acoustic fluid (rectangular room) and the compound absorber is carried out with the help of impedances. They are calculated for a unit load with spatially varying sinusoidal load patterns, described by the wavenumbers k_y and k_z (with respect to the global coordinate system), and the circular frequencies of excitation Ω. In figure 4 the result for an impedance of a compound absorber consisting of two layers of a linear elastic isotropic material (thickness $9.5\,mm$) filled with a porous medium (thickness $12\,cm$) is sketched exemplarily.



Compound absorber — Sectional view $k_z = 0$

Figure 4: Wavenumber dependent impedance of the compound absorber for $\Omega = 300\,\left[\frac{rad}{s}\right]$

Coupled System

The equilibrium is established for the coupled system in a weak form applying Hamilton´s Principle. Considering all deformation patterns, which are kinematically possible, the system will vibrate in the pattern, for which the integral of the difference from kinetic and potential energy over time has got a minimum. Non conservative forces (e.g. damping or external loads) are taken into account by their virtual work. The formulation of the Lagrangian function $\widetilde{L}_L$ is required.

The kinetic energy T_L and the potential energy U_L of our system, the rectangular acoustic volume (component 1), are defined as:

$$T_L = \tfrac{1}{2}\,\rho_L \int_V |\mathbf{v}_L(x,y,z,t)|^2\; dV$$

$$U_L = \tfrac{1}{2}\,\frac{1}{\rho_L\,c_L^2} \int_V p(x,y,z,t)^2\; dV$$

An efficient computation is possible, taking the orthogonality of the trigonometric functions in the normal modes and constraint modes in equation (2) into account. The Lagrangian function for component 1 finally results in $\widetilde{L}_L = T_L - U_L$.

Due to the fact, that there are no shear stresses carried by the air, absorber and acoustic volume are coupled only perpendicular to the boundaries (walls). Nevertheless these perpendicular movements cause damping effects in the porous medium because of pressure differences at the surface. In order to simplify the coupling equations, an approach for the surface-velocity v_x of the absorber is used, which corresponds to the spatial distribution of the constraint modes in y- and z-direction.

$$v_x^{+inf} = \sum_{r=0}^{r_{max}} \sum_{s=0}^{s_{max}} C_{rs}\,\cos\left(\frac{r\,\pi}{L_y}y\right)\,\cos\left(\frac{s\,\pi}{L_z}z\right)\,e^{(i\Omega t)} \quad (9)$$

Thus the interface constraints can be simply satisfied by arranging the coefficients ($B_{rs} = C_{rs}$), if the maximum number of wavenumbers (in both directions) in equation (9) is adapted to the maximum number of constraint modes considered. The wavenumber dependent impedance of the compound absorber is calculated assuming an infinite structure in y- and z-direction. For computing the Lagrangian function of the absorber the finite geometry of the subsystem has to be considered, defining a filter function (which is scaled with the geometry of the compound absorber), applying a Fourier series to the filtered surface velocity v_x^{inf} of the absorber and calculating the Lagrangian function out of the complex Fourier coefficients D_{rs}.
The wavenumber- and frequency-dependent impedance, calculated above, is not unique [4]. Generally two different structures can result in the same impedance. Although the structures are different, they behave equivalent. This means, that an application of a pressure p on each system will result in the same velocity v at the surface, where the pressure is applied. However these systems might not have identical kinetic and potential energy for each time t. The difference of kinetic and potential energy nevertheless is equal [4]. Therefore the Lagrangian function for our system (compound absorber) under a harmonic pressure load can be deduced from the imaginary part of the impedance $Z_{TPM}(k_y, k_z, \Omega)$ for each set of a circular frequency of excitation Ω and the wavenumbers k_y and k_z in y- and z-direction.

$$\widetilde{L}_{TPM} = \frac{1}{2}\,L_y^{rep}\,L_z^{rep} \sum_{-r_{max}^{FS}}^{r_{max}^{FS}} \sum_{-s_{max}^{FS}}^{s_{max}^{FS}} \frac{1}{\Omega}\,D_{rs}\,\overline{D}_{rs}\,Im\left(Z_{TPM}\right)$$

By means of the real part of $Z_{TPM}(k_y, k_z, \Omega)$ the mean power, which is put into the subsystem of the absorber and so the virtual work, done by the non conservative damping forces is gained.

$$W_{TPM}^{nc} = L_y^{rep}\,L_z^{rep} \sum_{-r_{max}^{FS}}^{r_{max}^{FS}} \sum_{-s_{max}^{FS}}^{s_{max}^{FS}} \frac{1}{i\,\Omega}\,D_{rs}\,\delta D_{rs}\,Re\left(Z_{TPM}\right)$$

For the computation of the Lagrangian function $\widetilde{L}_{TPM}$ and the virtual work W_{TPM}^{nc} of the subsystem (compound

absorber) considering the ansatz for the surface-velocity v_x in the finite system, sums of sets of Ω, k_y and k_z have to be taken into account, whereas each Foruier coefficient D_{rs} consists of a linear combination of coefficients C_{rs} in equation (9). Due to the orthogonality of the trigonometric functions in the Fourier series this can be carried out easily.
The external load is considered in Hamilton´s Principle as the virtual work of a non conservative force.

$$W_{Load}^{nc} = \int_0^{L_z}\int_0^{L_y} \frac{1}{i\,\Omega}\, p\,(x,y,z,t)\,\delta v_x\,(x,y,z,t)\;dy\,dz$$

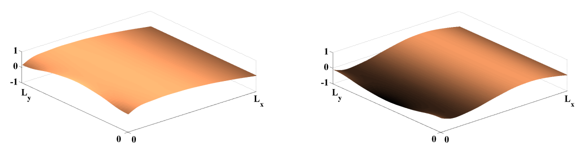
The Lagrangian function for the coupled system finally results in:

$$\delta \int_{t_1}^{t_2} \underbrace{\left(\widetilde{L}_L + \widetilde{L}_{TPM}\right)}_{\widetilde{L}^{tot}} dt + \delta \int_{t_1}^{t_2} W_{TPM}^{nc} - W_{Load}^{nc}\, dt = 0$$

The coefficients describing the contribution of the individual normal and constraint modes $A_{l\,m\,n}$ and $B_{m\,n}$ are computed out of a system of linear equations, gained applying Hamilton´s Principle taking account of the Ritz approach.

2D Example

In the following the method is applied for a 2D example. The system (Figure 1) is computed assuming plane waves in z-direction. The rectangular 2-dimensional room has got a length of $L_x = 6m$ in x-direction and $L_y = 2m$ in y-direction. Figure 5 shows exemplarily the first and the second modeshape of the room with reflective walls and a lightweight gypsum board wall fixed at $x = 0$.



First modeshape Second modeshape

Figure 5: Modeshapes for the fluid velocity v_x in x-direction of the rectangular room with an impedance boundary condition

In the following plane waves are excited in the system, applying a harmonically oscillating pressure load in x-direction, which is constant over L_y. The steady-state sound pressure is averaged over the room in order to compare a room with reflective walls and one with a compound absorber, consisting of two layers of a homogeneous elastic isotropic material covering a porous medium, at $x = 0$.

In Figure 6 one can see the dissipative effect, caused by the compound absorber, which results only from friction between the solid- and the gas-phase in the porous medium. There is no material damping defined in the homogeneous plates.

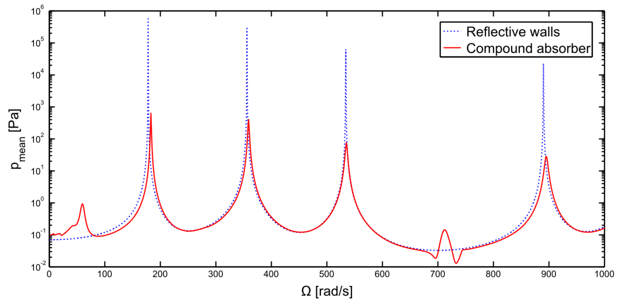


Figure 6: Averaged sound pressure p

Conclusions

A method for the computation of the steady state response of an acoustic cavity with a compound absorber, modeled as a plate-like structure consisting of homogeneous and porous layers, as an impedance boundary condition under a harmonic oscillating pressure load is presented. The wavenumber- and frequency-dependent impedances are computed using an Integral Transform Method. The method is based on a Ritz approach using fixed interface normal modes and constraint modes for the fluid. An application for more complex geometries is possible, evaluating the normal and constraint modes with the help of numerical methods (e.g. finite element or spectral element methods). The calculation of steady state responses for arbitrary wavenumbers and frequencies of excitation is possible. Thus it can be used by means of an inverse Fourier transform in order to calculate impulse response functions for applications in room acoustics.

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