Acoustic Streaming in a Viscous Fluid-Structure System

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Introduction

The existense and relevance of Acoustic Streaming (AS) for the mechanical processes in hearing has remained controversal. Main contributions to this topic came from the Bell Telephone Laboratories in 1972 [1], the work of Tonndorf and later from Lighthill [2]. Today a general assumption is the relevance of AS to be limited to very high sound pressure levels larger than 120 dB (SPL). Because of the unaccessability of the organ of Corti in the inner ear an experimental proof was not possible up to now, except the pioneering work of G. v. Békésy who saw eddies in the lymph of the cochlea with indeed high stimulating pressure levels (140 dB(SPL)). Therefore the actual opinion of AS in the ear ranges from it's irrelevance to it's permanent presence (pers.comm. Ch.Steele, 2008). AS is a physical phenomenon which was first proposed by Lord Rayleigh [3]. AS denotes sound induced flow (streaming) in fluids, e.g. air or water. While the mathematical foundation of the opposite effect, namely flow induced sound, was given by A. D. Pierce in the mid 1980s the development of the mathematical theory of AS is more vigorously and connected with the work of Lighthill [4]. Later it was extended by a "boundary drive" mechanism [5]. Köster [6] applies these theories to micro-fluidic mixing devices and provides a software for numerical evaluations, but the fluid-structure interaction was not yet implied. Rayleigh's law of streaming is applicable to two kinds of streaming motions

1. AS, resulting when an acoustic standing wave in a fluid adjacent to a solid wall suffers dissipation within the resulting boundary layer and
2. a related kind of streaming which results from the vibrations of a solid body adjacent to fluid at rest.

Because the sound dissipation and generation and the wave propagation in the cochlea of the inner ear of humans and animals are unsolved problems up to now we develope a mathematical theory which implements the fluid-structure interaction of the complex biomechanical system for a numerical evaluation.

Methods

In this section a technique is proposed to simulate acoustic streaming in a fluid-structure-coupled system like the cochlea. The system of equations that describes the fluid flow consists of the conversation relations of mass and momentum with the shear viscosity μ, bulk viscosity μ_B and density of the fluid ρ [5]

$$-\nabla p + \mu\nabla^2 \boldsymbol{v} + (\mu_B + \frac{\mu}{3})\nabla\nabla\cdot\boldsymbol{v} = \rho(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0 \tag{2}$$

and the constitutive relation $\rho = \rho(p)$. At the boundaries the no-slip condition is assumed. Since the acoustic streaming problem is a nonlinear effect the usual perturbation expansion of the unknowns $\boldsymbol{v}$, p and ρ is performed:

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{0} + \boldsymbol{v}^{(1)} + \boldsymbol{v}^{(2)} + \mathcal{O}(\epsilon^3) \\ p &= p^{(0)} + p^{(1)} + p^{(2)} + \mathcal{O}(\epsilon^3) \\ \rho &= \rho^{(0)} + \rho^{(1)} + \rho^{(2)} + \mathcal{O}(\epsilon^3) \end{aligned} \tag{3}$$

By combining equations (1), (2) and (3) and gathering only the linear terms the equations of mass and momentum become

$$-\nabla p^{(1)} + \mu\nabla^2\boldsymbol{v}^{(1)} + (\mu_B + \frac{\mu}{3})\nabla\nabla\cdot\boldsymbol{v}^{(1)} = \rho^{(0)}\frac{\partial \boldsymbol{v}^{(1)}}{\partial t} \tag{4}$$

$$\frac{1}{c_0^2\rho^{(0)}}\frac{\partial p^{(1)}}{\partial t} + \nabla\cdot\boldsymbol{v}^{(1)} = \boldsymbol{0} \tag{5}$$

where $p^{(1)} = c_0^2\rho^{(1)}$ is assumed. The first order system is the standard linear system which describes the damped propagation of sound in a viscous fluid. The boundary condition that describes the coupling between the fluid and the structure is given by

$$\boldsymbol{v} = \frac{\partial \boldsymbol{u}}{\partial t} \tag{6}$$

where $\boldsymbol{u}$ denotes the displacement of the structure. Applying a finite element discretisation of the acoustic system (4) and (5) with simultaneous consideration of the coupling condition (6) yields

$$\begin{aligned} \boldsymbol{A}\dot{\tilde{\boldsymbol{v}}} + \boldsymbol{B}\tilde{\boldsymbol{v}} + \boldsymbol{D}\tilde{\boldsymbol{p}} + \boldsymbol{E}\dot{\tilde{\boldsymbol{u}}} &= \boldsymbol{0} \\ \boldsymbol{F}\dot{\tilde{\boldsymbol{p}}} + \boldsymbol{G}\tilde{\boldsymbol{p}} + \boldsymbol{H}\tilde{\boldsymbol{v}} + \boldsymbol{I}\dot{\tilde{\boldsymbol{u}}} &= \boldsymbol{0} \end{aligned} \tag{7}$$

The discrete structural problem is given by [7]

$$\boldsymbol{M}\ddot{\tilde{\boldsymbol{u}}} + \boldsymbol{C}\dot{\tilde{\boldsymbol{u}}} + \boldsymbol{K}\tilde{\boldsymbol{u}} + \boldsymbol{Q}\tilde{\boldsymbol{p}} = \boldsymbol{f} \tag{8}$$

By combining (7) and (8) the linear fluid-structure coupled acoustic system becomes

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M} \end{bmatrix} \begin{pmatrix} \ddot{\tilde{\boldsymbol{v}}} \\ \ddot{\tilde{\boldsymbol{p}}} \\ \ddot{\tilde{\boldsymbol{u}}} \end{pmatrix} + \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{E} \\ \boldsymbol{0} & \boldsymbol{F} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{C} \end{bmatrix} \begin{pmatrix} \dot{\tilde{\boldsymbol{v}}} \\ \dot{\tilde{\boldsymbol{p}}} \\ \dot{\tilde{\boldsymbol{u}}} \end{pmatrix} \\ &+ \begin{bmatrix} \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{H} & \boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} & \boldsymbol{K} \end{bmatrix} \begin{pmatrix} \tilde{\boldsymbol{v}} \\ \tilde{\boldsymbol{p}} \\ \tilde{\boldsymbol{u}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{f} \end{pmatrix} \end{aligned} \tag{9}$$

Equation (8) contains the equation of motion of the elastic-embedded beam which is captured by space dependent finite elements of the stiffness matrix $\mathbf{K}$ easily.

The solutions of the linear velocity, pressure and displacement field are used for the time-averaged second order system of equations of mass and momentum, given by

$$\nabla p^{(2)} - \mu \nabla^2 \boldsymbol{v}^{(2)} - (\mu_B + \frac{\mu}{3}) \nabla\nabla \cdot \boldsymbol{v}^{(2)} = -\frac{1}{c_0^2} \langle p^{(1)} \frac{\partial \boldsymbol{v}^{(1)}}{\partial t} \rangle - \rho^{(0)} \langle (\boldsymbol{v}^{(1)} \cdot \nabla) \boldsymbol{v}^{(1)} \rangle \qquad (10)$$

$$\rho^{(0)} \nabla \cdot \boldsymbol{v}^{(2)} = -\frac{1}{c_0^2} \nabla \cdot \langle p^{(1)} \boldsymbol{v}^{(1)} \rangle \qquad (11)$$

where $\langle \cdot \rangle$ denotes the temporal average of a time-dependent function. Note, that the second harmonic terms are also time-averaged, since the time-independent part of the fluid-motion is of interest. Finally the acoustic streaming field can be derived by an appropriate finite element discretisation of the second order system (10).

Figure 1 shows the 2D model of the cochlea including the fluid-structure coupling which will be examined first. The shaded area marks the outer boundary (bone) which encloses the lymph. The upper left vertical line represents the oval window and the lower left vertical line the round window. The horizontal line marks the basilar membrane (BM) which carries the sensory cells responsible for acoustic-neural transduction (inner and outer hair cells) necessary for hearing sensations but neglected in a preliminary passive model with the BM represented by an elastic-embedded beam.

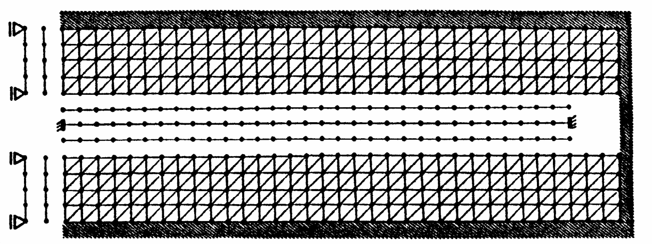

Figure 1: 2D Finite Element model of the cochlea

Figure 2 shows the push-pull displacement of the oval window and round window and the BM [1]. The extremely exaggerated displacements sketches the acoustical wave on the BM. Especially with higher frequencies ($f > 2000Hz$) the push-pull mechanism of the oval and round window vanishes because of the lymph's compressibility. This might lead to in-phase movements of the windows.

The calculation of the primary acoustic field (Equations 4, 5) is time consuming especially without parallel processing. The use of the Boundary Element Method (BEM) brought about a relevant gain but could be implemented by Köster only for 2D problems up to now. The effective solution of the coupled system (9) is under development.

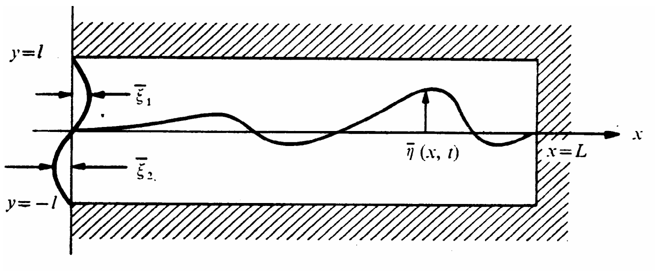

Figure 2: Sketch of exaggerated displacements of cochlear structures: oval window, round window and basilar membrane (see [1], L = 35 mm)

Results

As a first step the acoustic wave propagation in a two-dimensional rectangle (32 mm x 2 mm) was computed with a vibrating boundary using the Finite Element Software ALBERTA. First numerical results of a time domain formulation indicated compressional waves in the fluid (lymph) propagating with the small-signal sound speed of water

$$c_0 = 1484 \, \frac{m}{s} \qquad (12)$$

In this case a Crank-Nicolson discretisation scheme in time and Lagrangian elements in space were used. Further calculations will be performed with the fully coupled system calculating the variables (pressure and displacements) of the first and second order system in two and three dimensions.

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