Fluid structure interaction and non-local admittance boundary conditions: Setup of an analytical example

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Introduction

In systems with fluid and structure coupled at their interfaces, the system equations are formulated in terms of variables for the fluid, e.g. sound pressure, and variables for the structure, e.g. displacement. The variables for the structure can be substituted by evaluation of the Schur complement. In that case, part of the Schur complement can be understood as a coupling admittance with non-local boundary admittance entries. Herein, the authors try to explain and emphasize the effect on that coupling admittance by changing this non-local to a local definition.
A one dimensional problem of a duct is introduced as a simple example in order to understand this issue by means of a boundary–element–method–like matrix formulation. Single degree of freedom systems are introducted at both ends of the duct. For the setup of an example with non-local boundary conditions, both ends are connected by two springs and a rigid block. For this system, the admittance matrix will be shown analytically and we will discuss certain cases, e.g. the case of a very heavy connecting block which makes the admittance condition a local one.

Local admittance

First of all we imagine a one–dimensional fluid domain bounded on one side by a flexible wall that may be considered as a simple mass–spring–damper system as shown in Figure 1.

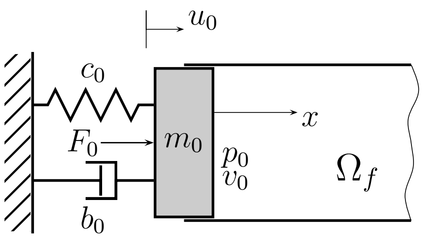

Figure 1: One–dimensional model of fluid bounded by mass–spring–damper system (MSD) as substitute for flexible wall.

For this system the balance of forces written in the time domain is

$$m_0\ddot{u}_0 + b_0\dot{u}_0 + c_0 u_0 = F_0 - Ap_0 \tag{1}$$

with A being the cross–sectional area of the interface. If we confine ourselves to the stationary state we may transform this in terms of the circular frequency ω

$$-\omega^2 m_0 u_0 - \mathrm{i}\omega b_0 u_0 + c_0 u_0 = F_0 - Ap_0\,. \tag{2}$$

Based on the Robin boundary condition formulation

$$v_0 - v_{0s} = Yp \tag{3}$$

with the structural velocity set to zero ($v_{0s} = 0$) and

$$v_0 = -\mathrm{i}\omega u_0 \tag{4}$$

we find the acoustical admittance

$$Y_0 = \frac{v_0}{p_0} = \frac{-\mathrm{i}\omega A}{-\omega^2 m_0 - \mathrm{i}\omega b_0 + c_0}\,. \tag{5}$$

During the following mathematics this parameter expresses the dynamics of the wall. It is a frequency dependend and complex parameter (Figure 2).

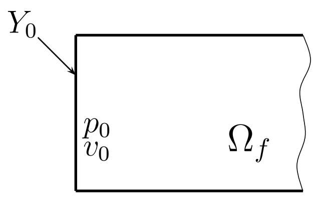

Figure 2: Acoustical admittance parameter Y_0 as equivalent for the MSD.

Discrete matrix formulation

The boundary value problem for a stationary vibrating system consists of the Lame–Navier equation for the structure, of the Helmholtz equation for the fluid and of the full coupling condition $v_f = v_s$ along the fluid structure interaction (FSI).
In the frequency domain we obtain this matrix formulation:

$$\begin{bmatrix} \boldsymbol{H} & -\boldsymbol{G}\boldsymbol{C}_{fs} \\ -\boldsymbol{C}_{sf} & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{f} \end{bmatrix}. \tag{6}$$

In this context $\boldsymbol{H}$ and $\boldsymbol{G}$ are system matrices from the boundary element formulation (BEM), matrix $\boldsymbol{A}$ derives from a finite element formulation (FEM). The matrices $\boldsymbol{C}_{fs}$ and $\boldsymbol{C}_{sf}$ allow to convert structural displacement and fluid velocity back and forth.
Column matrix $\boldsymbol{p}$ denotes the nodal sound pressure

amplitude, $\boldsymbol{u}$ the displacement of the structure.
Applying Schur complement eliminates $\boldsymbol{u}$

$$\left(\boldsymbol{H} - \boldsymbol{G}\boldsymbol{C}_{fs}\boldsymbol{A}^{-1}\boldsymbol{C}_{sf}\right)\boldsymbol{p} = \boldsymbol{G}\boldsymbol{C}_{fs}\boldsymbol{A}^{-1}\boldsymbol{f}\,. \qquad (7)$$

Here we may introduces the coupling admittance matrix

$$\boldsymbol{Y}_c = \boldsymbol{C}_{fs}\boldsymbol{A}^{-1}\boldsymbol{C}_{sf}\,. \qquad (8)$$

Thus we obtain a matrix equation reduced in size to give the sound pressure as solution:

$$\left(\boldsymbol{H} - \boldsymbol{G}\boldsymbol{Y}_c\right)\boldsymbol{p} = \boldsymbol{G}\boldsymbol{v} \qquad \text{with} \qquad \boldsymbol{v} = \boldsymbol{C}_{fs}\boldsymbol{A}^{-1}\boldsymbol{f}\,. \quad (9)$$

$\boldsymbol{Y}_c$ is generally densely occupied due to full coupling.

Meaning of non-local admittance

In this section the one–dimensional, analytical problem of a coupled structure fluid system is put into a particular form of a system of equation that matches equation (7) and equation (9) respectively.
Figure 3 shows this structural–fluid system with mass spring damper systems enclosing the fluid on either side. The two systems are themselves connected dynamically by an additional mass m_b. Therewith the structure acts in a non–local fashion towards the fluid.

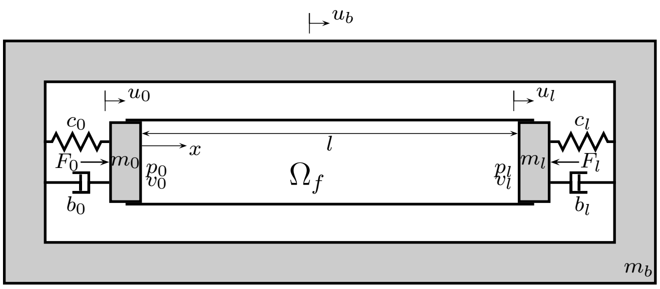

Figure 3: One–dimensional fluid with MSDs on either side and an additional mass m_b.

As shown in Figure 3 we introduced a couple of degrees of freedom on the interfaces between the fluid and the two masses m_0 and m_l: the sound pressures p_0 and p_l as well as the normal fluid velocities v_0 and v_l.
In the following we shall neglect a detailed description of the necessary mathematical steps. Instead, they shall be accounted for verbally.
Firstly, we assemble the balances of forces on the three masses m_0, m_l and m_b for their displacements u_0, u_l and u_b. The balance on m_b may then be used to remove u_b. Applying relation (4) in the frequency domain, the other two displacements are replaced by their interface velocities v_0 and v_l.
The analytical one–dimensional solution of the Helmholtz equation provides us with two equations for the sound pressures p_0 and p_l.
Coupling structure and fluid implies merging the four remaining equations to this system of equation:

$$\begin{bmatrix} h_{11} & h_{12} & -\mathrm{i}\rho c\omega & 0 \\ h_{21} & h_{22} & 0 & \mathrm{i}\rho c\omega \\ A & 0 & d_0 - \frac{w_0^2}{d_b} & -\frac{w_0 w_l}{d_b} \\ 0 & -A & -\frac{w_0 w_l}{d_b} & d_l - \frac{w_l^2}{d_b} \end{bmatrix} \begin{bmatrix} p_0 \\ p_l \\ u_0 \\ u_l \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_0 \\ -F_l \end{bmatrix} \qquad (10)$$

Applying the Schur complement again breaks this system down to an equation that matches equation (9).
Now, here in particular the coupling admittance matrix looks like this:

$$\boldsymbol{Y}_c = \frac{-\mathrm{i}\omega A}{N} \begin{bmatrix} d_l - \frac{w_l^2}{d_b} & -\frac{w_0 w_l}{d_b} \\ -\frac{w_0 w_l}{d_b} & d_0 - \frac{w_0^2}{d_b} \end{bmatrix}\,. \qquad (11)$$

The constants h_{ij}, d_0, d_l, d_b, w_0, w_l und N appearing in equation (10) and (11) are as follows:

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \frac{e^{\mathrm{i}kl} + e^{-\mathrm{i}kl}}{e^{\mathrm{i}kl} - e^{-\mathrm{i}kl}} & -\frac{2}{e^{\mathrm{i}kl} - e^{-\mathrm{i}kl}} \\ -\frac{2}{e^{\mathrm{i}kl} - e^{-\mathrm{i}kl}} & \frac{e^{\mathrm{i}kl} + e^{-\mathrm{i}kl}}{e^{\mathrm{i}kl} - e^{-\mathrm{i}kl}} \end{bmatrix} \qquad (12)$$

and

$$\begin{aligned} d_0 &= -\omega^2 m_0 - \mathrm{i}\omega b_0 + c_0 \\ d_l &= -\omega^2 m_l - \mathrm{i}\omega b_l + c_l \\ d_b &= -\omega^2 m_b - \mathrm{i}\omega\left(b_0 + b_l\right) + \left(c_0 + c_l\right) \\ w_0 &= -\mathrm{i}\omega b_0 + c_0 \\ w_l &= -\mathrm{i}\omega b_l + c_l \\ N &= d_0 d_l - \frac{d_0 w_l^2}{d_b} - \frac{d_l w_0^2}{d_b}\,. \end{aligned} \qquad (13)$$

As with a coupling admittance of any complex model discretized using FEM and BEM, $\boldsymbol{Y}_c$ is fully occupied. In order to understand the effect of a non-local and a local structural behavior on the equations we imagine an increasing mass m_b. If m_b reaches infinity, the movement of the two masses m_0 and m_l become then uncoupled, i.e. the dynamics the structural path is cut.
Looking at the coupling admittance in (11) we realize that the matrix becomes diagonal. Therefore, the essence of this contribution is: a local definition of admittance boundary condition within acoustical calculations leads to a diagonalization of the coupling admittance matrix. It is important for any simulation applications to investigate whether this fact may turn out advantagerous or disadvantagerous.

References

[1] D. Fritze, S. Marburg, and H.-J. Hardtke. FEM–BEM–coupling and structural–acoustic sensitivity analysis for shell geometries. *Computers and Structures*, 83:143–154, 2005.