An approach for stable calculation of vocal fold oscillation

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Abstract

The modeling of continuous string-like systems is often performed by mechanical models using a system of coupled masses [1, 2] (see Figure 1 below). Mathematically this discrete mechanical system leads to a system of coupled ordinary differential equations (o.d.e.'s) which can be solved efficiently by numerical schemes. In order to increase the model's accuracy, it would be interesting to increase the spatial resolution of the mechanical model by adding more masses. However, in many cases this leads to numerical instabilities. In this paper we propose a mechanically natural, mathematically simple and numerically stable and efficient solution to this problem: the discrete mechanical model is transformed into a continuum model by considering infinitely many, infinitely fine masses. Mathematically this corresponds to passing from the ode system to a p.d.e. (partial differential equation). Numerically, the resulting nonlinear wave-equation is solved stably if one controls the time-step by a Courant-Friedrichs-Lewy (CFL) condition [3].

1 Introduction

The vocal fold can be approximately described by a system of coupled masses. In common models there is one layer of masses representing the vocalis and another layer representing the mucosa. A two-mass-model uses only one mass per vocal fold for each layer, and these layers are coupled by an appropriate spring. One of the first two mass models was generated by Ishizaka and Flanagan [4]. A 16-mass-model was introduced by Titze [1, 2], who divided each of the two layers into eight masses along the vocal fold. A lengthwise discretisation postulates a new force, the tension-force, that acts on the masses and reflects the elasticity of the layers. When coupled lengthwise by this force, the 16 masses constitute a string-like character of the formation.

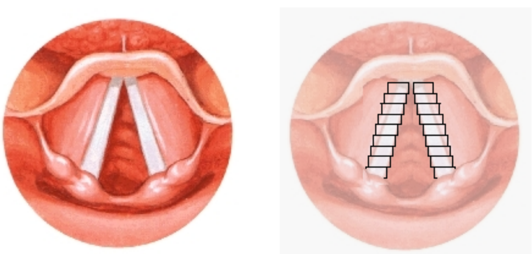


Figure 1: spacial discretisation of the vocal folds [5]

A further degree of freedom was introduced by Kob [6], who developed models with a variable number of lengthwise masses, allowing for arbitrary spatial discretisations. But Kob himself observed that such arbitrary discretisations may lead to numerical instabilities while computing the vocal fold oscillation.

2 Method

In the discrete models the movement of each mass is calculated from Newton's Law

$$m \cdot \ddot{x} \quad = \quad \sum F(x), \tag{1}$$

where x is the position of the mass with weight m and $\sum F(x)$ the sum of the acting forces which depend on the position x. Hence the movement is the solution of this o.d.e. system. Since the discrete mechanical model [6] allows an arbitrary number of masses, it can be transformed into a continuum model by considering infinitely many, infinitely fine masses. Mathematically this corresponds to passing from the o.d.e. system to a p.d.e.. The resulting p.d.e is a nonlinear wave-equation that can be solved stably, if one controls the time-step by a CFL condition.

An algorithm observing this condition was implemented in JAVA to compute the movement of the vocal folds.

3 From a multi mass model to a non-linear wave equation

In order to identify the small scales in the o.d.e. system, we begin by writing it in dimensionless form.

First we consider the multi mass model and focus on the tension force F^T, which reflects the elasticity of the layers, and on oscillations in the x-direction. Let

$$\ddot{x}_i \quad = \quad \gamma \cdot F_x^T(x_i, x_{i-1}, x_{i+1}) \tag{2}$$

be the non-dimensional Newton law, where F_x^T is the x-component of the tension force for an arbitrary mass x_i. The index i indicates the lengthwise position of the mass. The tension force F_x^T is calculated as

$$\begin{aligned} F_x^T = \quad & F^T(x_i, x_{i+1}) \cdot \frac{x_{i+1} - x_i}{\sqrt{a^2 + (x_{i+1} - x_i)^2}} \\ & - F^T(x_{i-1}, x_i) \cdot \frac{x_i - x_{i-1}}{\sqrt{a^2 + (x_i - x_{i-1})^2}}, \end{aligned} \tag{3}$$

where

$$F^T(x_i, x_{i+1}) = F^T\left(-1 + \sqrt{1 + \frac{(x_{i+1} - x_i)^2}{a^2}}.\right) \quad (4)$$

describes the (modulus of the) non-dimensional force resulting from the strain and a is the non-dimensional distance between the center-points of two neighbouring masses. For sufficiently many masses a is small, while due to the non-dimensionalisation the other parameters are of order 1. Therefore, the approximation

$$\frac{\partial x_{i+\frac{1}{2}}}{\partial y} \approx \frac{x_{i+1} - x_i}{a} \quad (5)$$

is accurate up to terms of order a^2. This allows to pass to the limit $a \to 0$ in a Taylor expansion of F^T. This yields the non-linear wave equation

$$\ddot{x} = \delta^2 \frac{\partial \left(F^T\left(\frac{\partial x}{\partial y}\right) \cdot \frac{\frac{\partial x}{\partial y}}{\sqrt{1+(\frac{\partial x}{\partial y})^2}} \right)}{\partial y}. \quad (6)$$

4 Mathematical properties of the wave equation

Let us pause for a moment and consider the non-linear wave equation (6). For linear right-hand-sides, it reads

$$x_{tt} = c^2 \cdot x_{yy} \quad (7)$$

with velocity $c > 0$. Its solution is given by D'Alembert's formula

$$u(y,t) = \frac{1}{2}[f(y + c \cdot t) + f(y - c \cdot t)] + \frac{1}{2}\int_{y-c\cdot t}^{y+c\cdot t} g(y)dy$$

where $f \in C^2(\mathbb{R})$ and $g \in C^2(\mathbb{R})$ are the initial values for the displacement and the velocity [7]. The formula shows that the propagation of the linear wave in the space-time-plane is cone-like with velocity c.

It has been shown by Lax (1964) [8] and Klainerman and Majda (1980) [9] that solutions to nonlinear wave equations develop singularities in finite time. Therefore, an initially smooth fold, which was governed by the tension force alone, would develop a kink. The only reason that this is not seen in models for the vocal fold is that other damping forces are present.

5 Implementation of a stable approach

An efficient and accurate difference scheme for the wave equation is derived from using the leapfrog-scheme ([10], p.159) twice. For the linear wave equation (7) this gives

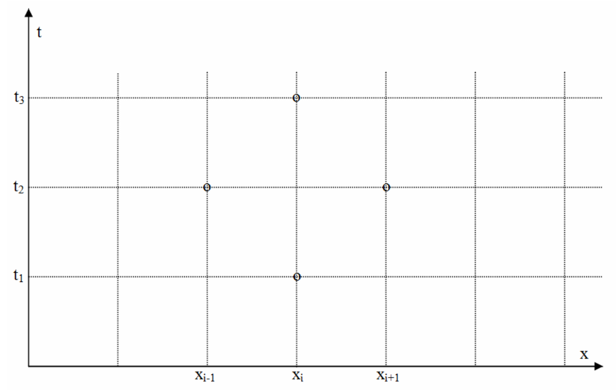
$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{(\Delta t)^2} = c^2 \cdot \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta y)^2}, \quad (8)$$

and for the nonlinear wave equation (6)

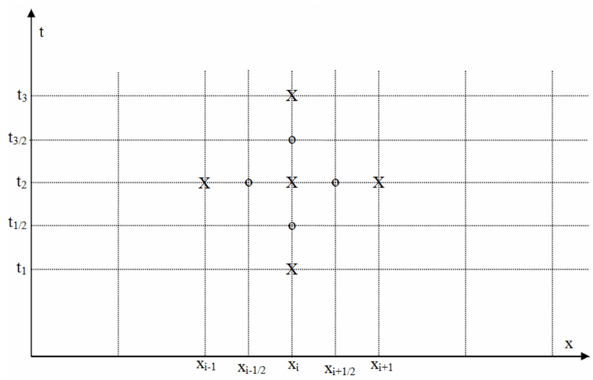
$$u_i^{j+1} = \delta^2 \frac{K(\frac{u_{i+1}^j - u_i^j}{\Delta y}) - K(\frac{u_i^j - u_{i-1}^j}{\Delta y})}{\Delta y} + 2u_i^j - u_i^{j-1}, \quad (9)$$

where

$$K(\frac{\partial x}{\partial y}) := F^T\left(-1 + \sqrt{1 + (\frac{\partial x}{\partial y})^2}\right) \cdot \frac{\frac{\partial x}{\partial y}}{\sqrt{1 + (\frac{\partial x}{\partial y})^2}}.$$



(a) leapfrog-scheme



(b) leapfrog-scheme twice

The stability of these schemes depends on the choice of time-step. Already in 1928 Courant, Friedrichs, Lewy [3] observed that the numerical domain of dependence of a convergent numerical scheme must always contain the physical domain of dependence. For the linear case this means that

$$\Delta t < \frac{\Delta y}{c}. \quad (10)$$

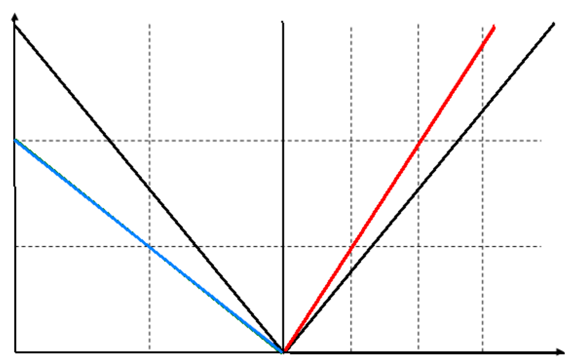


Figure 2: Propagation of the linear wave

For the nonlinear case, on must replace c in (10) in every timestep by δ^2 times the maximal derivative of K over x_i. Thus $\Delta t = \Delta t_j$. The resulting scheme is remarkably stable and second order accurate in space and time.

6 Discussion

The continuum model can be used for synthesis of normal and pathologic voices. A comparison of the algorithm introduced in this paper and an ordinary algorithm will be done soon.

Implementations of a simplified vocal fold model have been made in MATLAB and JAVA. The result is tested with a multi-mass-model of vocal fold oscillation. It is shown that for an arbitrary number of masses a stable calculation is obtained. The application of the method for more complex models of normal and pathologic voice production is planned.

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